3 Conservation of Mechanical Energy II: Springs, Rotational Kinetic Energy

A common mistake involving springs is using the length of a stretched spring when the amount of stretch is called for. Given the length of a stretched spring, you have to subtract off the length of that same spring when it is neither stretched nor compressed to get the amount of stretch.

Spring Potential Energy is the potential energy stored in a spring that is compressed or stretched. The spring energy depends on how stiff the spring is and how much it is stretched or compressed. The stiffness of the spring is characterized by the *force constant* of the spring, k. k is also referred to as the *spring constant* for the spring. The stiffer the spring, the bigger its value of k is. The symbol x is typically used to characterize the amount by which a spring is compressed or stretched. It is important to note that x is not the length of the stretched or compressed spring. Instead, it is the difference between the length of the stretched or compressed spring and the length of the spring when it is neither stretched nor compressed. The amount of energy U_S stored in a spring with a force constant (spring constant) k that has either been stretched by an amount x or compressed by an amount x is:

$$U_s = \tfrac{1}{2} k x^2 \tag{3-1}$$

Rotational Kinetic Energy is the energy that a spinning object has because it is spinning. When an object is spinning, every bit of matter making up the object is moving in a circle (except for those bits on the axis of rotation). Thus, every bit of matter making up the object has some kinetic energy $\frac{1}{2} m v^2$ where the v is the speed of the bit of matter in question and m is its mass. The thing is, in the case of an object that is just spinning, the object itself is not going anywhere, so it has no speed, and the different bits of mass making up the object have different speeds, so there is no one speed v that we can use for the speed of the object in our old expression for kinetic energy $K = \frac{1}{2} m v^2$. The amount of kinetic energy that an object has because it is spinning can be expressed as:

$$K = \tfrac{1}{2} \mathrm{I}\, \omega^2 \tag{3-2}$$

where the Greek letter omega ω (please don't call it double-u) is used to represent the magnitude of the angular velocity of the object and the symbol I is used to represent the moment of inertia, a.k.a. rotational inertia, of the object. The magnitude of the angular velocity of the object is how fast the object is spinning and the moment of inertia of the object is a measure of the object's natural tendency to spin at a constant rate. The greater the moment of inertia of an object, the harder it is to change how fast that object is spinning.

The magnitude of the angular velocity, the spin rate ω, is measured in units of radians per second where the radian is a unit of angle. An angle is a fraction of a rotation and hence a unit of angle is a fraction of a rotation. If we divide a rotation up into 360 parts then each part is $\dfrac{1}{360}$ of a rotation and we call each part a degree. In the case of radian measure, we divide the rotation up into 2π parts and call each part a radian. Thus a radian is $\frac{1}{2\pi}$ of a rotation. The fact that an angle is a fraction of a rotation means that an angle is really a pure number and the word "radian" abbreviated rad, is a reminder about how many parts the rotation has been divided up into, rather than a true unit. In working out the units in cases involving radians, one can simply erase the word radian. This is not the case for actual units such as meters or joules.

The moment of inertia I has units of $\mathrm{kg \cdot m^2}$. The units of the right hand side of equation 3-2, $K = \frac{1}{2}\mathrm{I}\,\omega^2$, thus work out to be $\mathrm{kg \cdot m^2 \frac{rad^2}{s^2}}$. Taking advantage of the fact that a radian is not a true unit, we can simply erase the units $\mathrm{rad^2}$ leaving us with units of $\mathrm{kg \cdot \frac{m^2}{s^2}}$, a combination that we recognize as a joule which it must be since the quantity on the left side of the equation $K = \frac{1}{2}\mathrm{I}\,\omega^2$ (equation 3-2) is an energy.

Energy of Rolling

An object which is rolling is both moving through space and spinning so it has both kinds of kinetic energy, the $\frac{1}{2}mv^2$ and the $\frac{1}{2}\mathrm{I}\,\omega^2$. The movement of an object through space is called translation. To contrast it with rotational kinetic energy, the ordinary kinetic energy $K = \frac{1}{2}mv^2$ is referred to as translational kinetic energy. So, the total kinetic energy of an object that is rolling can be expressed as

$$K_{Rolling} = K_{Translation} + K_{Rotation} \tag{3-3}$$

$$K_{Rolling} = \frac{1}{2}mv^2 + \frac{1}{2}\mathrm{I}\,\omega^2 \tag{3-4}$$

Now you probably recognize that an object that is rolling without slipping is spinning at a rate that depends on how fast it is going forward. That is to say that the value of ω depends on the value of v. Let's see how. When an object that is rolling without slipping completes one rotation, it moves a distance equal to its circumference which is 2π times the radius of that part of the object on which the object is rolling.

$$\text{Distance traveled in one rotation} = 2\pi r \tag{3-5}$$

Now if we divide both sides of this equation by the amount of time that it takes for the object to complete one rotation we obtain on the left, the speed of the object and, on the right, we can interpret the 2π as 2π radians and, since 2π radians is one rotation the 2π radians divided by the time it takes for the object to complete one rotation is just the magnitude of the angular velocity ω. Hence we arrive at

$$v = \omega r$$

which is typically written:

$$v = r\omega \tag{3-6}$$