4 Conservation of Momentum

A common mistake involving conservation of momentum crops up in the case of totally inelastic collisions of two objects, the kind of collision in which the two colliding objects stick together and move off as one. The mistake is to use conservation of mechanical energy rather than conservation of momentum. One way to recognize that some mechanical energy is converted to other forms is to imagine a spring to be in between the two colliding objects such that the objects compress the spring. Then imagine that, just when the spring is at maximum compression, the two objects become latched together. The two objects move off together as one as in the case of a typical totally inelastic collision. After the collision, there is energy stored in the compressed spring so it is clear that the total kinetic energy of the latched pair is less than the total kinetic energy of the pair prior to the collision. There is no spring in a typical inelastic collision. The mechanical energy that would be stored in the spring, if there was one, results in permanent deformation and a temperature increase of the objects involved in the collision.

The momentum of an object is a measure of how hard it is to stop that object. The momentum of an object depends on both its mass and its velocity. Consider two objects of the *same mass*, e.g. two baseballs. One of them is coming at you at 10 mph, and the other at 100 mph. Which one has the greater momentum? Answer: The *faster* baseball is, of course, harder to stop, so it has the greater momentum. Now consider two objects of *different mass* with the *same velocity*, e.g. a Ping-Pong ball and a cannon ball, both coming at you at 25 mph. Which one has the greater momentum? The cannon ball is, of course, harder to stop, so it has the greater momentum.

The momentum p of an object is equal to the product[1] of the object's mass m and velocity v:

$$p = mv \tag{4-1}$$

Momentum has direction. Its direction is the same as that of the velocity. In this chapter we will limit ourselves to motion along a line (motion in one dimension). Then there are only two directions, forward and backward. An object moving forward has a positive velocity/momentum and one moving backward has a negative velocity/momentum. In solving physics problems, the decision as to which way is forward is typically left to the problem solver. Once the problem solver decides which direction is the positive direction, she must state what her choice is (this statement, often made by means of notation in a sketch, is an important part of the solution), and stick with it throughout the problem.

The concept of momentum is important in physics because the total momentum of any system remains constant unless there is a net transfer of momentum to that system, and if there is an ongoing momentum transfer, the rate of change of the momentum of the system is equal to the rate at which momentum is being transferred into the system. As in the case of energy, this means that one can make predictions regarding the outcome of physical processes by means of

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[1] This classical physics expression is valid for speeds small compared to the speed of light $c = 3.00\times10^8$m/s. The relativistic expression for momentum is $p = mv\Big/\sqrt{1-v^2/c^2}$. At speeds that are very small compared to the speed of light, the classical physics expression $p = mv$ is a fantastic approximation to the relativistic expression.

simple accounting (bookkeeping) procedures. The case of momentum is complicated by the fact that momentum has direction, but in this initial encounter with the conservation of momentum you will deal with cases involving motion along a straight line. When all the motion is along one and the same line, there are only two possible directions for the momentum and we can use algebraic signs (plus and minus) to distinguish between the two. The principle of Conservation of Momentum applies in general. At this stage in the course however, we will consider only the special case in which there is no net transfer of momentum to (or from) the system from outside the system.

Conservation of Momentum in One Dimension for the Special Case in which there is No Transfer of Momentum to or from the System from Outside the System

In any process involving a system of objects which all move along one and the same line, as long as none of the objects are pushed or pulled along the line by anything outside the system of objects (it's okay if they push and pull on each other), the total momentum before, during, and after the process remains the same.

The total momentum of a system of objects is just the algebraic sum of the momenta of the individual objects. That adjective "algebraic" means you have to pay careful attention to the plus and minus signs. If you define "to the right" as your positive direction and your system of objects consists of two objects, one moving to the right with a momentum of 12 kg·m/s and the other moving to the left with momentum 5 kg·m/s, then the total momentum is (+12 kg·m/s) + (−5 kg·m/s) which is +7 kg·m/s. The plus sign in the final answer means that the total momentum is directed to the right.

Upon reading this selection you'll be expected to be able to apply conservation of momentum to two different kinds of processes. In each of these two classes of processes, the system of objects will consist of only two objects. In one class, called *collisions*, the two objects bump into each other. In the other class, *anti-collisions* the two objects start out together, and spring apart. Some further breakdown of the collisions class is pertinent before we get into examples. The two extreme types of collisions are the *completely **inelastic** collision*, and the *completely **elastic** collision*.

Upon a *completely inelastic collision*, the two objects stick together and move off as one. This is the easy case since there is only one final velocity (because they are stuck together, the two objects obviously move off at one and the same velocity). Some mechanical energy is converted to other forms in the case of a completely inelastic collision. It would be a big mistake to apply the principle of conservation of mechanical energy to a completely inelastic collision. *Mechanical energy is **not** conserved*. The words "completely inelastic" tell you that both objects have the same velocity (as each other) after the collision.

In a *completely **elastic** collision* (often referred to simply as an ***elastic** collision*), the objects bounce off each other in such a manner that no mechanical energy is converted into other forms in the collision. Since the two objects move off independently after the collision there are two final velocities. If the masses and the initial velocities are given, conservation of momentum yields one equation with two unknowns—namely, the two final velocities. Such an equation cannot be solved by itself. In such a case, one must apply the principle of conservation of mechanical energy. It does apply here. The expression "completely elastic" tells you that conservation of mechanical energy does apply.

In applying conservation of momentum one first sketches a before and an after picture in which one defines symbols by labeling objects and arrows (indicating velocity), and defines which direction is chosen as the positive direction. The first line in the solution is always a statement that the total momentum in the before picture is the same as the total momentum in the after picture. This is typically written by means an equation of the form:

$$\sum p_{\rightarrow} = \sum p'_{\rightarrow} \qquad (4\text{-}2)$$

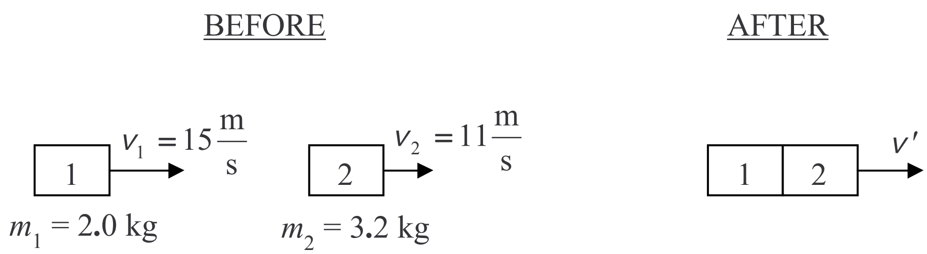
The Σ in this expression is the upper case Greek letter "sigma" and is to be read "the sum of." Hence the equation reads: "The sum of the momenta to the right in the before picture is equal to the sum of the momenta to the right in the after picture." In doing the sum, a leftward momentum counts as a negative rightward momentum. The arrow subscript is being used to define the positive direction.

Examples

Now let's get down to some examples. We'll use the examples to clarify what is meant by collisions and anti-collisions; to introduce one more concept, namely, relative velocity (sometimes referred to as muzzle velocity); and of course, to show the reader how to apply conservation of momentum.

Example 4-1

Two objects move on a horizontal frictionless surface along the same line in the same direction which we shall refer to as the forward direction. The trailing object of mass 2.0 kg has a velocity of 15 m/s forward. The leading object of mass 3.2 kg has a velocity of 11 m/s forward. The trailing object catches up with the leading object and the two objects experience a completely inelastic collision. What is the final velocity of each of the two objects?

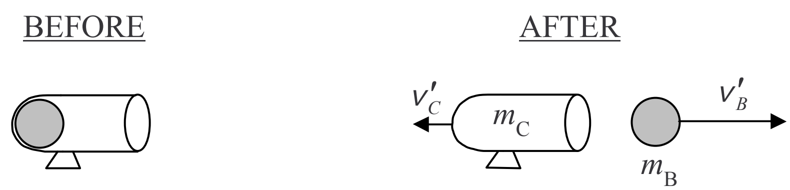


$$\Sigma p_{\rightarrow} = \Sigma p'_{\rightarrow}$$
$$p_1 + p_2 = p'_{12}$$
$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v'$$
$$v' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$
$$v' = \frac{2.0\,\text{kg}\,(15\,\text{m/s}) + 3.2\,\text{kg}\,(11\,\text{m/s})}{2.0\,\text{kg} + 3.2\,\text{kg}}$$
$$v' = 12.54\,\frac{\text{m}}{\text{s}}$$
$$v' = 13\,\frac{\text{m}}{\text{s}}$$

The final velocity of each of the objects is $\boxed{13\,\frac{\text{m}}{\text{s}}\text{ forward.}}$

Example 4-2: A cannon of mass m_C, resting on a frictionless surface, fires a ball of mass m_B. The ball is fired horizontally. The muzzle velocity is v_M. Find the velocity of the ball and the recoil velocity of the cannon.

NOTE: This is an example of an anti-collision problem. It also involves the concept of relative velocity. The muzzle velocity is the relative velocity between the ball and the cannon. It is the velocity at which the two separate. If the velocity of the ball relative to the ground is v'_B to the right, and the velocity of the cannon relative to the ground is v'_C to the left, then the velocity of the ball relative to the cannon, also known as the muzzle velocity of the ball, is $v_M = v'_B + v'_C$. In cases not involving guns or cannons one typically uses the notation v_{rel} for "relative velocity" or, relating to the example at hand, v_{BC} for "velocity of the ball relative to the cannon."



$$\Sigma p_{\rightarrow} = \Sigma p'_{\rightarrow}$$

$$0 = -m_C v'_C + m_B v'_B \qquad (1)$$

Also, from the definition of muzzle velocity:

$$v_M = v'_B + v'_C$$

$$v'_C = v_M - v'_B \qquad (2)$$

Substituting this result into equation (1) yields:

$$0 = -m_C(v_M - v'_B) + m_B v'_B$$

$$0 = -m_C v_M + m_C v'_B + m_B v'_B$$

$$m_C v'_B + m_B v'_B = m_C v_M$$

$$(m_C + m_B) v'_B = m_C v_M$$

$$\boxed{v'_B = \frac{m_C}{m_C + m_B} v_M}$$

Now substitute this result into equation (2) above. This yields:

$$v'_C = v_M - \frac{m_C}{m_C + m_B} v_M$$

$$v'_C = \frac{(m_C + m_B)v_M - m_C v_M}{m_C + m_B}$$

$$v'_C = \frac{m_C v_M + m_B v_M - m_C v_M}{m_C + m_B}$$

$$\boxed{v'_C = \frac{m_B}{m_C + m_B} v_M}$$