5 Conservation of Angular Momentum

Much as in the case of linear momentum, the mistake that tends to be made in the case of angular momentum is not using the principle of conservation of angular momentum when it should be used, that is, applying conservation of mechanical energy in a case in which mechanical energy is not conserved but angular momentum is. Consider the case, for instance, in which one drops a disk (from a negligible height) that is not spinning, onto a disk that is spinning, and after the drop, the two disks spin together as one. The "together as one" part tips you off that this is a completely inelastic (rotational) collision. Some mechanical energy is converted into thermal energy (and other forms not accounted for) in the collision. It's easy to see that mechanical energy is converted into thermal energy if the two disks are CD's and the bottom one is initially spinning quite fast (but is not being driven). When you drop the top one onto the bottom one, there will be quite a bit of slipping before the top disk gets up to speed and the two disks spin as one. During the slipping, it is friction that increases the spin rate of the top CD and slows the bottom one. Friction converts mechanical energy into thermal energy. Hence, the mechanical energy prior to the drop is less than the mechanical energy after the drop.

The angular momentum of an object is a measure of how difficult it is to stop that object from spinning. For an object rotating about a fixed axis, the angular momentum depends on how fast the object is spinning, and on the object's *rotational inertia* (also known as *moment of inertia*) with respect to that axis.

Rotational Inertia (a.k.a. Moment of Inertia)

The rotational inertia of an object with respect to a given rotation axis is a measure of the object's tendency to resist a change in its angular velocity about that axis. The rotational inertia depends on the mass of the object and how that mass is distributed. You have probably noticed that it is easier to start a merry-go-round spinning when it has no children on it. When the kids climb on, the mass of what you are trying to spin is greater, and this means the rotational inertia of the object you are trying to spin is greater. Have you also noticed that if the kids move in toward the center of the merry-go-round it is easier to start it spinning than it is when they all sit on the outer edge of the merry-go-round? It is. The farther, on the average, the mass of an object is distributed away from the axis of rotation, the greater the object's moment of inertia with respect to that axis of rotation. The rotational inertia of an object is represented by the symbol I. During this initial coverage of angular momentum, you will not be required to calculate I from the shape and mass of the object. You will either be given I or expected to calculate it by applying conservation of angular momentum (discussed below).

Angular Velocity

The angular velocity of an object is a measure of how fast it is spinning. It is represented by the Greek letter omega, written ω, (not to be confused with the letter w which, unlike omega, is pointed on the bottom). The most convenient measure of angle in discussing rotational motion is the radian. Like the degree, a radian is a fraction of a revolution. But, while one degree is $\frac{1}{360}$ of a revolution, one radian is $\frac{1}{2\pi}$ of a revolution. The units of angular velocity are then *radians per second* or, in notational form, $\frac{\text{rad}}{\text{s}}$. Angular velocity has direction or sense of rotation associated with it. If one defines a rotation which is clockwise when viewed from above as a positive rotation, then an object which is rotating counterclockwise as viewed from above is said to have a negative angular velocity. In any problem involving angular velocity, one is free to choose the positive sense of rotation, but then one must stick with that choice throughout the problem.

Angular Momentum

The angular momentum L of an object is given by:

$$L = \text{I}\omega \tag{5-1}$$

Note that this is consistent with our original definition of angular momentum as a measure of the degree of the object's tendency to keep on spinning, once it is spinning. The greater the rotational inertia of the object, the more difficult it is to stop the object from spinning, and the greater the angular velocity of the object, the more difficult it is to stop the object from spinning.

The direction of angular momentum is the same as the direction of the corresponding angular velocity.

Torque

We define torque by analogy with force which is an ongoing push or pull on an object. When there is a single force acting on a particle, the momentum of that particle is changing. A torque is what you are exerting on the lid of a jar when you are trying to remove the lid. When there is a single torque acting on a rigid object, the angular momentum of that object is changing.

Conservation of Angular Momentum

Angular Momentum is an important concept because, if there is no angular momentum transferred to or from a system, the total angular momentum of that system does not change, and if there is angular momentum being transferred to a system, the rate of change of the angular momentum of the system is equal to the rate at which angular momentum is being transferred to the system. As in the case of energy and momentum, this means we can use simple accounting (bookkeeping) procedures for making predictions on the outcomes of physical processes. In this chapter we focus on the special case in which there are no external torques which means that no angular momentum is transferred to or from the system.

Conservation of Angular Momentum for the Special Case in which no Angular Momentum is Transferred to or from the System from Outside the System

In any physical process involving an object or a system of objects free to rotate about an axis, as long as there are no external torques exerted on the system of objects, the total angular momentum of that system of objects remains the same throughout the process.

Examples

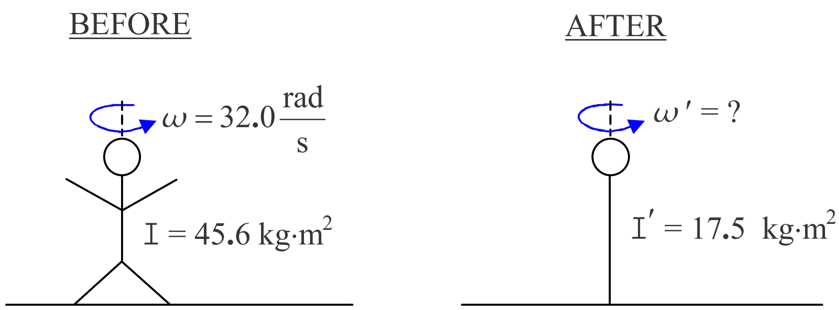
The application of the conservation of angular momentum in solving physics problems for cases involving no transfer of angular momentum to or from the system from outside the system (no external torque) is very similar to the application of the conservation of energy and to the application of the conservation of momentum. One selects two instants in time, defines the earlier one as the before instant and the later one as the after instant, and makes corresponding sketches of the object or objects in the system. Then one writes

$$L = L' \tag{5-2}$$

meaning "the angular momentum in the before picture equals the angular momentum in the after picture." Next, one replaces each L with what it is in terms of the moments of inertia and angular velocities in the problem and solves the resulting algebraic equation for whatever is sought.

Example 5-1

A skater is spinning at 32.0 rad/s with her arms and legs extended outward. In this position her moment of inertia with respect to the vertical axis about which she is spinning is $45.6\,\mathrm{kg\cdot m^2}$. She pulls her arms and legs in close to her body changing her moment of inertia to $17.5\,\mathrm{kg\cdot m^2}$. What is her new angular velocity?



$$L_{\circlearrowright} = L'_{\circlearrowright}$$

$$I\omega = I'\omega'$$

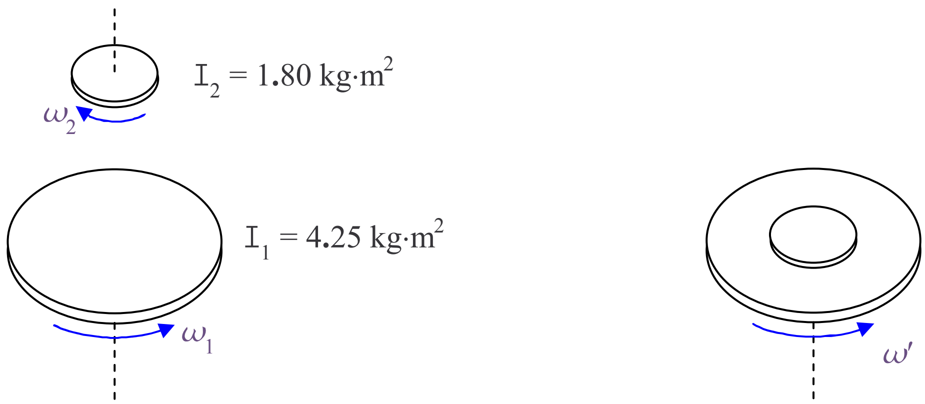
$$\omega' = \frac{I}{I'}\omega$$

$$\omega' = \frac{45.6\,\mathrm{kg\cdot m^2}}{17.5\,\mathrm{kg\cdot m^2}}\,32.0\,\mathrm{rad/s}$$

$$\boxed{\omega' = 83.4\,\frac{\mathrm{rad}}{\mathrm{s}}}$$

Example 5-2

A horizontal disk of rotational inertia $4.25\,\text{kg}\cdot\text{m}^2$ with respect to its axis of symmetry is spinning counterclockwise about its axis of symmetry, as viewed from above, at 15.5 revolutions per second on a frictionless massless bearing. A second disk, of rotational inertia $1.80\,\text{kg}\cdot\text{m}^2$ with respect to its axis of symmetry, spinning *clockwise* as viewed from above about the same axis (which is also its axis of symmetry) at 14.2 revolutions per second, is dropped on top of the first disk. The two disks stick together and rotate as one about their common axis of symmetry at what new angular velocity (in units of radians per second)?



Some preliminary work (expressing the given angular velocities in units of rad/s):

$$\omega_1 = 15.5\,\frac{\text{rev}}{\text{s}}\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right) = 97.39\,\frac{\text{rad}}{\text{s}} \qquad \omega_2 = 14.2\,\frac{\text{rev}}{\text{s}}\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right) = 89.22\,\frac{\text{rad}}{\text{s}}$$

Now we apply the principle of conservation of angular momentum for the special case in which there is no transfer of angular momentum to or from the system from outside the system. Referring to the diagram:

$$L_{\circlearrowleft} = L'_{\circlearrowleft}$$

We define counterclockwise, as viewed from above, to be the "+" sense of rotation.

$$I_1\,\omega_1 - I_2\,\omega_2 = (I_1 + I_2)\,\omega'$$

$$\omega' = \frac{I_1\,\omega_1 - I_2\,\omega_2}{I_1 + I_2}$$

$$\omega' = \frac{(4.25\,\text{kg}\cdot\text{m}^2)\,97.39\,\text{rad/s} - (1.80\,\text{kg}\cdot\text{m}^2)\,89.22\,\text{rad/s}}{4.25\,\text{kg}\cdot\text{m}^2 + 1.80\,\text{kg}\cdot\text{m}^2}$$

$$\boxed{\omega' = 41.9\,\frac{\text{rad}}{\text{s}}}$$ (Counterclockwise as viewed from above.)