# 7 One-Dimensional Motion: The Constant Acceleration Equations

*The constant acceleration equations presented in this chapter are only applicable to situations in which the acceleration is constant. The most common mistake involving the constant acceleration equations is using them when the acceleration is changing.*

In chapter 6 we established that, by definition,

$$a = \frac{dv}{dt}$$

(which we called equation 6-5) where $a$ is the acceleration of an object moving along a straight line path, $v$ is the velocity of the object and $t$, which stands for time, represents the reading of a stopwatch.

This equation is called a differential equation because that is the name that we give to equations involving derivatives. It's true for any function that gives a value of $a$ for each value of $t$. An important special case is the case in which $a$ is simply a constant. Here we derive some relations between the variables of motion for just that special case, the case in which $a$ is constant.

Equation 6-5, $a = \frac{dv}{dt}$, with $a$ stipulated to be a constant, can be considered to be a relationship between $v$ and $t$. Solving it is equivalent to finding an expression for the function that gives the value of $v$ for each value of $t$. So our goal is to find the function whose derivative $\frac{dv}{dt}$ is a constant. The derivative, with respect to $t$, of a constant times $t$ is just the constant. Recalling that we want that constant to be $a$, let's try:

$$v = at$$

We'll call this our trial solution. Let's plug it into equation 6-5, $a = \frac{dv}{dt}$, and see if it works. Equation 6-5 can be written:

$$a = \frac{d}{dt}v$$

and when we plug our trial solution $v = at$ into it we get:

$$a = \frac{d}{dt}(at)$$

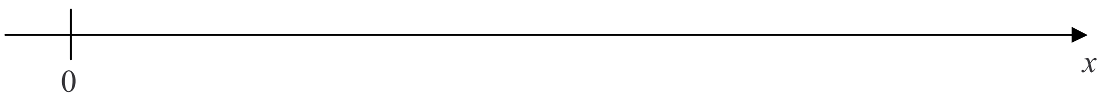
$$a = a\frac{d}{dt}t$$

$$a = a \cdot 1$$

$$a = a$$

That is, our trial solution $v = at$ leads to an identity. Thus, our trial solution is indeed a solution to the equation $a = \frac{dv}{dt}$. Let's see how this solution fits in with the linear motion situation under study.

In that situation, we have an object moving along a straight line and we have defined a one-dimensional coordinate system which can be depicted as




and consists of nothing more than an origin and a positive direction for the position variable $x$. We imagine that someone starts a stopwatch at a time that we define to be "time zero," $t = 0$, a time that we also refer to as "the start of observations." Rather than limit ourselves to the special case of an object that is at rest at the origin at time zero, we assume that it could be moving with any velocity and be at any position on the line at time zero and define the constant $x_o$ to be the position of the object at time zero and the constant $v_o$ to be the velocity of the object at time zero.

Now the solution $v = at$ to the differential equation $a = \frac{dv}{dt}$ yields the value $v = 0$ when $t = 0$ (just plug $t = 0$ into $v = at$ to see this). So, while $v = at$ does solve $a = \frac{dv}{dt}$, it does not meet the conditions at time zero, namely that $v = v_o$ at time zero. We can fix the initial condition problem easily enough by simply adding $v_o$ to the original solution yielding

$$v = v_o + at \tag{7-1}$$

This certainly makes it so that $v$ evaluates to $v_o$ when $t = 0$. But is it still a solution to $a = \frac{dv}{dt}$? Let's try it. If $v = v_o + at$, then

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(v_o + at\right) = \frac{d}{dt}v_o + \frac{d}{dt}(at) = 0 + a\frac{d}{dt}t = a \quad .$$

$v = v_o + at$, when substituted into $a = \frac{dv}{dt}$ leads to an identity so $v = v_o + at$ is a solution to $a = \frac{dv}{dt}$. What we have done is to take advantage of the fact that the derivative of a constant is zero, so if you add a constant to a function, you do not change the derivative of that function. The solution $v = v_o + at$ is not only a solution to the equation $a = \frac{dv}{dt}$ (with $a$ stipulated to be a constant) but it is a solution to the whole problem since it also meets the initial value condition that $v = v_o$ at time zero. The solution, equation 7-1:

$$v = v_o + at$$

is the first of a set of four constant acceleration equations to be developed in this chapter.

The other definition provided in the last section was equation 6-2:

$$v = \frac{dx}{dt}$$

which in words can be read as: The velocity of an object is the rate of change of the position of the object (since the derivative of the position with respect to time is the rate of change of the position). Substituting our recently-found expression for velocity yields

$$v_o + at = \frac{dx}{dt}$$

which can be written as:

$$\frac{dx}{dt} = v_o + at \qquad (7\text{-}2)$$

We seek a function that gives a value of $x$ for every value of $t$, whose derivative $\frac{dx}{dt}$ is the sum of terms $v_o + at$. Given the fact that the derivative of a sum will yield a sum of terms, namely the sum of the derivatives, let's try a function represented by the expression $x = x_1 + x_2$. This works if $\frac{dx_1}{dt}$ is $v_o$ and $\frac{dx_2}{dt}$ is $at$. Let's focus on $x_1$ first. Recall that $v_o$ is a constant. Further recall that the derivative-with-respect-to-$t$ of a constant times $t$, yields that constant. So check out $x_1 = v_o t$. Sure enough, the derivative of $v_o t$ with respect to $t$ is $v_o$, the first term in equation 7-2 above. So far we have

$$x = v_o t + x_2 \qquad (7\text{-}3)$$

Now let's work on $x_2$. We need $\frac{dx_2}{dt}$ to be $at$. Knowing that when we take the derivative of something with $t^2$ in it we get something with $t$ in it we try $x_2 = \text{constant} \cdot t^2$. The derivative of that is $2 \cdot \text{constant} \cdot t$ which is equal to $at$ if we choose $\frac{1}{2}a$ for the constant. If the constant is $\frac{1}{2}a$ then our trial solution for $x_2$ is $x_2 = \frac{1}{2}at^2$. Plugging this in for $x_2$ in equation 7-3, $x = v_o t + x_2$, yields:

$$x = v_o t + \frac{1}{2}at^2$$

Now we are in a situation similar to the one we were in with our first expression for $v(t)$. This expression for $x$ *does* solve

$$\frac{dx}{dt} = v_o + at \tag{7-4}$$

but it does *not* give $x_o$ when you plug 0 in for $t$. Again, we take advantage of the fact that you can add a constant to a function without changing the derivative of that function. This time we add the constant $x_o$ so

$$x = x_o + v_o t + \frac{1}{2}at^2 \tag{7-5}$$

This meets both our criteria: It solves equation 7-4, $\frac{dx}{dt} = v_o + at$, and it evaluates to $x_o$ when $t = 0$. We have arrived at the second equation in our set of four constant acceleration equations.

The two that we have so far are, equation 7-5:

$$x = x_o + v_o t + \frac{1}{2}at^2$$

and equation 7-1:

$$v = v_o + at$$

These two are enough, but to simplify the solution of constant acceleration problems, we use algebra to come up with two more constant acceleration equations. Solving equation 7-1, $v = v_o + at$, for $a$ yields $a = \frac{v - v_o}{t}$ and if you substitute that into equation 7-5 you quickly arrive at the third constant acceleration equation

$$x = x_o + \frac{v_o + v}{2}t \tag{7-6}$$

Solving equation 7-1, $v = v_o + at$, for $t$ yields $t = \frac{v - v_o}{a}$ and if you substitute that into equation 7-5 you quickly arrive at the final constant acceleration equation:

$$v^2 = v_o^2 + 2a(x - x_o) \tag{7-7}$$

For your convenience, we copy down the entire set of constant acceleration equations that you are expected to use in your solutions to problems involving constant acceleration:

$$x = x_o + v_o t + \frac{1}{2}at^2$$

$$x = x_o + \frac{v_o + v}{2}t$$

$$v = v_o + at$$

$$v^2 = v_o^2 + 2a(x - x_o)$$

(*Constant Acceleration Equations*)