17 The Universal Law of Gravitation

Consider an object released from rest an entire moon's diameter above the surface of the moon. Suppose you are asked to calculate the speed with which the object hits the moon. *This problem typifies the kind of problem in which students use the universal law of gravitation to get the force exerted on the object by the gravitational field of the moon, and then mistakenly use one or more of the constant acceleration equations to get the final velocity. The problem is: the acceleration is not constant. The closer the object gets to the moon, the greater the gravitational force, and hence, the greater the acceleration of the object. The mistake lies not in using Newton's second law to determine the acceleration of the object at a particular point in space; the mistake lies in using that one value of acceleration, good for one object-to-moon distance, as if it were valid on the entire path of the object. The way to go on a problem like this, is to use conservation of energy.*

Back in chapter 12, where we discussed the near-surface gravitational field of the earth, we talked about the fact that any object that has mass creates an invisible force-per-mass field in the region of space around it. We called it a gravitational field. Here we talk about it in more detail. Recall that when we say that an object causes a gravitational field to exist, we mean that it creates an invisible force-per-mass vector at every point in the region of space around itself. The effect of the gravitational field on any particle (call it the victim) that finds itself in the region of space where the gravitational field exists, is to exert a force, on the victim, equal to the force-per-mass vector at the victim's location, times the mass of the victim.

Now we provide a quantitative discussion of the gravitational field. (*Quantitative* means, involving formulas, calculations, and perhaps numbers. Contrast this with *qualitative* which means descriptive/conceptual.) We start with the idealized notion of a point particle of matter. Being matter, it has mass. Since it has mass it has a gravitational field in the region around it. The direction of a particle's gravitational field at point P, a distance r away from the particle, is *toward the particle* and the magnitude of the gravitational field is given by

$$g = G\frac{m}{r^2} \tag{17-1}$$

where:

G is the universal gravitational constant: $G = 6.67 \times 10^{-11}\,\dfrac{\text{N}\cdot\text{m}^2}{\text{kg}^2}$

m is the mass of the particle, and
r is the distance that point P is from the particle.

In that point P can be any empty (or occupied) point in space whatsoever, this formula gives the magnitude of the gravitational field of the particle at all points in space. Equation 17-1 is the equation form of Newton's Universal Law of Gravitation[1].

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[1] Newton's Universal Law of Gravitation is an approximation to Einstein's far more complicated General Theory of Relativity. The approximation is fantastic for orbital mechanics for space vehicles and most planets, but we need to use the General Theory of Relativity for a complete explanation of the orbit of Mercury.

The Total Gravitational Field Vector at an Empty Point in Space

Suppose that you have two particles. Each has its own gravitational field at all points in space. Let's consider a single empty point in space. Each of the two particles has its own gravitational field vector at that empty point in space. We can say that each particle makes its own vector contribution to the total gravitational field at the empty point in space in question. So how do you determine the total gravitational field at the empty point in space? You guessed it! Just add the individual contributions. And because the contribution due to each particle is a vector, the two contributions add like vectors[2].

What the Gravitational Field does to a Particle that is in the Gravitational Field

Now suppose that you have the magnitude and direction of the gravitational field vector $\vec{g}$ at a particular point in space. The gravitational field exerts a force on any "victim" particle that happens to find itself at that location in space. Suppose, for instance, that a particle of mass m finds itself at a point in space where the gravitational field (of some other particle or particles) is $\vec{g}$. Then the particle of mass m is subject to a force

$$\vec{F}_G = m\vec{g} \qquad (17\text{-}2)$$

The Gravitational Effect of one Particle on Another

Let's put the two preceding ideas together. Particle 1 of mass m_1 has, among its infinite set of gravitational field vectors, a gravitational field vector at a location a distance r_{12} away from itself, a point in space that happens to be occupied by another particle, particle 2, of mass m_2. The gravitational field of particle 1 exerts a force on particle 2. The question is, how big and which way is the force?

Let's start by identifying the location of particle 2 as point P. Point P is a distance r_{12} away from particle 1. Thus, the magnitude of the gravitational field vector (of particle 1) at point P is:

$$g = G\frac{m_1}{r_{12}^2} \qquad (17\text{-}3)$$

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[2] The statement that the various contributions to the gravitational field add like vectors is known as the superposition principle for the gravitational field. Einstein, in his General Theory of Relativity, showed that the superposition principle for the gravitational field is actually an approximation that is excellent for the gravitational fields you will deal with in this course but breaks down in the case of very strong gravitational fields.

Now the force exerted on particle 2 by the gravitational field of particle 1 is given by equation 17-2, $\vec{F}_G = m_2 \vec{g}$. Using $\vec{F}_{12}$ for $\vec{F}_G$ to emphasize the fact that we are talking about the force of 1 on 2, and writing the equation relating the magnitudes of the vectors we have

$$F_{12} = m_2 g$$

Replacing the g with the expression we just found for the magnitude of the gravitational field due to particle number 1 we have

$$F_{12} = m_2\, G \frac{m_1}{r_{12}^2}$$

which, with some minor reordering can be written as

$$F_{12} = G \frac{m_1\, m_2}{r_{12}^{\ 2}} \qquad\qquad (17\text{-}4)$$

This equation gives the force of the gravitational field of particle 1 on particle 2. Neglecting the "middleman" (the gravitational field) we can think of this as the force of particle 1 on particle 2. You can go through the whole argument again, with the roles of the particles reversed, to find that the same expression applies to the force of particle 2 on particle 1, or you can simply invoke Newton's 3rd Law to arrive at the same conclusion.

Objects, Rather than Point Particles

The vector sum of all the gravitational field vectors due to a spherically symmetric distribution of point particles (for instance, a spherically symmetric solid object), at a point outside the distribution (e.g. outside the object), is the same as the gravitational field vector due to a single particle, at the center of the distribution, whose mass is equal to the sum of the masses of all the particles. Also, for purposes of calculating the force exerted by the gravitational field of a point object on a spherically symmetric victim, one can treat the victim as a point object at the center of the victim. Finally, regarding either object in a calculation of the gravitational force exerted on a rigid object by the another object: if the separation of the objects is very large compared to the dimensions of the object, one can treat the object as a point particle located at the center of mass of the object and having the same mass as the object. This goes for the gravitational potential energy, discussed below, as well.

How Does this Fit in with $g = 9.80$ N/kg?

When we talked about the earth's near-surface gravitational field before, we used a single value for its magnitude, namely 9.80 N/kg and said that it always directed downward, toward the center of the earth. 9.80 N/kg is actually an average sea level value. g varies within about a half a percent of that value over the surface of the earth and the handbook value includes a tiny centrifugal pseudo force-per-mass field contribution (affecting both magnitude and direction) stemming from the rotation of the earth. So how does the value 9.80 N/kg for the magnitude of the gravitational field near the surface of the earth relate to Newton's Universal Law of Gravitation?

Certainly the direction is consistent with our understanding of what it should be: The earth is roughly spherically symmetric so for purposes of calculating the gravitational field outside of the earth we can treat the earth as a point particle located at the center of the earth. The direction of the gravitational field of a point particle is toward that point particle, so, anywhere outside the earth, including at any point *just* outside the earth (near the surface of the earth), the gravitational field, according to the Universal Law of Gravitation, must be directed toward the center of the earth, a direction we earthlings call downward.

But how about the magnitude? Shouldn't it vary with elevation according to the Universal Law of Gravitation? First off, how does the magnitude calculated using $g = G\dfrac{m}{r^2}$ compare with 9.80 N/kg at, for instance, sea level. A point at sea level is a distance $r = 6.37 \times 10^6$ m from the center of the earth. The mass of the earth is $m = 5.98 \times 10^{21}$ kg. Substituting these values into our expression for g (equation 17-4 which reads $g = G\dfrac{m_1}{r_{12}^2}$) we find:

$$g = 6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2} \frac{5.98 \times 10^{24} \text{ kg}}{(6.37 \times 10^6 \text{ m})^2}$$

$$g = 9.83 \frac{\text{N}}{\text{kg}}$$

which does agree with the value of 9.80 N/kg to within about 0.3 percent. (The difference is due in part to the centrifugal pseudo force-per-mass field associated with the earth's rotation.) We can actually see, just from the way that the value of the radius of the earth, 6.37×10^6 m, is written, that increasing our elevation, even by a mile (1.61 km), is not going to change the value of g to three significant figures. We'd be increasing r from 6.37×10^6 m to 6.37161×10^6 m which, to three significant figures is still 6.37×10^6 m. This brings up the question, "How high above the surface do you have to go before $g = 9.80 \dfrac{\text{N}}{\text{kg}}$ is no longer within a certain percentage of the value obtained by means of the Universal Law of Gravitation?" Let's investigate that question for the case of a 1 percent difference. In other words, how high above sea level would

we have to go to make g = 99% of g at sea level, that is g = (.99)(9.80 m/s^2) = 9.70 m/s^2. Letting $r = r_E + h$ where r_E is the radius of the earth, and using g = 9.70 N/kg so that we can find the h that makes 9.70 N/kg we have:

$$g = \frac{Gm_E}{(r_E + h)^2}$$

Solving this for h yields

$$h = \sqrt{\frac{Gm_E}{g}} - r_E$$

$$h = \sqrt{\frac{6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2} 5.98 \times 10^{24} \text{ kg}}{9.70 \frac{N}{kg}}} - 6.37 \times 10^6 \text{ m}$$

$$h = .04 \times 10^6 \text{ m}$$

That is to say that at any altitude above 40 km above the surface of the earth, you should use Newton's Universal Law of Gravitation directly in order for your results to be good to within 1%.

In summary, g = 9.80 N/kg for the near-earth-surface gravitational field magnitude is an approximation to the Universal Law of Gravitation good to within about 1% anywhere within about 40 km of the surface of the earth. In that region, the value is approximately a constant because changes in elevation represent such a tiny fraction of the total earth's-center-to-surface distance as to be negligible.

The Universal Gravitational Potential Energy

So far in this course you have become familiar with two kinds of potential energy, the near-earth's-surface gravitational potential energy $U_g = mgy$ and the spring potential energy $U_S = \frac{1}{2}kx^2$. Here we introduce another expression for gravitational potential energy. This one is pertinent to situations for which the Universal Law of Gravitation is appropriate.

$$U_G = -\frac{Gm_1 m_2}{r_{12}} \tag{17-5}$$

This is the gravitational potential energy of a pair of particles, one of mass m_1 and the other of mass m_2, which are separated by a distance of r_{12}. Note that for a given pair of particles, the gravitational potential energy can take on values from negative infinity up to zero. Zero is the highest possible value and it is the value of the gravitational potential energy at infinite separation. That is to say that $U_G \xrightarrow[\lim r_{12} \to \infty]{} 0$. The lowest conceivable value is negative infinity and it would be the value of the gravitational potential energy of the pair of particles if one could put them so close together that they were both at the same point in space. In mathematical notation: $U_G \xrightarrow[\lim r_{12} \to 0]{} -\infty$.

Conservation of Mechanical Energy Problems Involving Universal Gravitational Potential Energy for the Special Case in which the Total Amount of Mechanical Energy does not Change

You solve fixed amount of mechanical energy problems involving universal gravitational potential energy just as you solved fixed amount of mechanical energy problems involving other forms of potential energy back in chapters 2 and 3. Draw a good before and after picture, write an equation stating that the energy in the before picture is equal to the energy in the after picture, and proceed from there. To review these procedures, check out the example problem on the next page:

Example 17-1: How great would the muzzle velocity of a gun on the surface of the moon have to be in order to shoot a bullet to an altitude of 101 km?

Solution: We'll need the following lunar data:

Mass of the moon: $m_m = 7.35 \times 10^{22}\,\text{kg}$;   Radius of the moon: $r_m = 1.74 \times 10^6\,\text{m}$
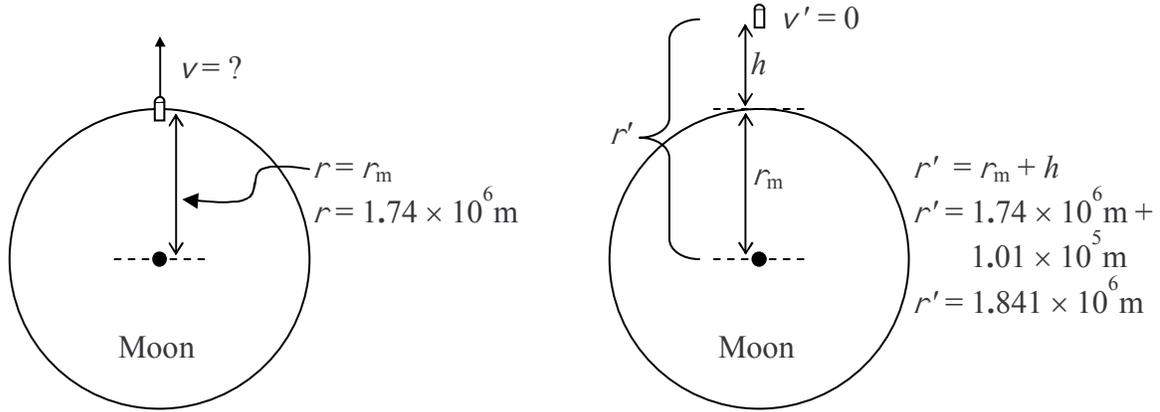
$$E = E'$$

$$K + U = K' + U'$$

$$\frac{1}{2}mv^2 + -\frac{Gm_m m}{r} = -\frac{Gm_m m}{r'}$$

(m_m is the mass of the moon and m is the mass of the bullet.)

$$\frac{1}{2}v^2 = \frac{Gm_m}{r} - \frac{Gm_m}{r'}$$

$$\frac{1}{2}v^2 = \frac{Gm_m}{r} - \frac{Gm_m}{r'}$$

$$\frac{1}{2}v^2 = Gm_m\left(\frac{1}{r} - \frac{1}{r'}\right)$$

$$v = \sqrt{2Gm_m\left(\frac{1}{r} - \frac{1}{r'}\right)}$$

$$v = \sqrt{2\left(6.67 \times 10^{-11}\,\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)7.35 \times 10^{22}\,\text{kg}\left(\frac{1}{1.74 \times 10^6\,\text{m}} - \frac{1}{1.841 \times 10^6\,\text{m}}\right)}$$

$$\boxed{v = 556\,\frac{\text{m}}{\text{s}}}$$