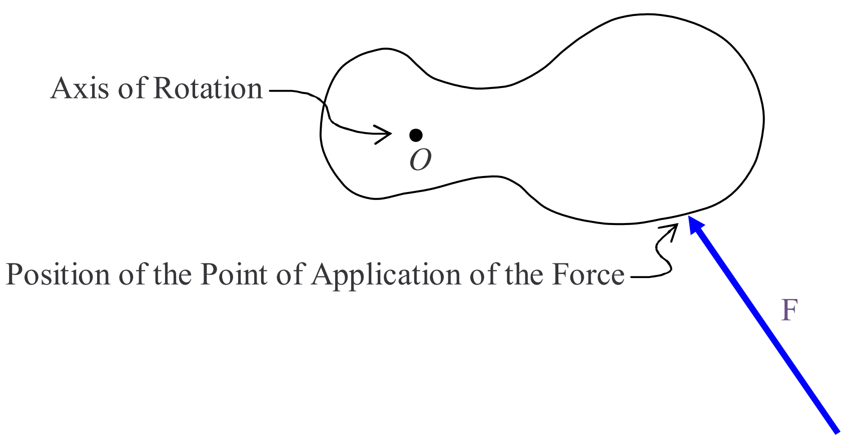
21 Vectors: The Cross Product & Torque

Do not use your left hand when applying either the right-hand rule for the cross product of two vectors (discussed in this chapter) or the right-hand rule for "something curly something straight" discussed in the preceding chapter.

There is a **relational operator**[1] for vectors that allows us to bypass the calculation of the moment arm. The relational operator is called the cross product. It is represented by the symbol "×" read "cross." The torque $\vec{\boldsymbol{\tau}}$ can be expressed as the cross product of the position vector $\vec{\mathbf{r}}$ for the point of application of the force, and the force vector $\vec{\mathbf{F}}$ itself:

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} \tag{21-1}$$

Before we begin our mathematical discussion of what we mean by the cross product, a few words about the vector $\vec{\mathbf{r}}$ are in order. It is important for you to be able to distinguish between the position vector $\vec{\mathbf{r}}$ for the force, and the moment arm, so we present them below in one and the same diagram. We use the same example that we have used before:



in which we are looking directly along the axis of rotation (so it looks like a dot) and the force lies in a plane perpendicular to that axis of rotation. We use the diagramatic convention that, the point at which the force is applied to the rigid body is the point at which one end of the arrow in the diagram touches the rigid body. Now we add the line of action of the force and the moment arm $r_\perp$ to the diagram, as well as the position vector $\vec{\mathbf{r}}$ of the point of application of the force.

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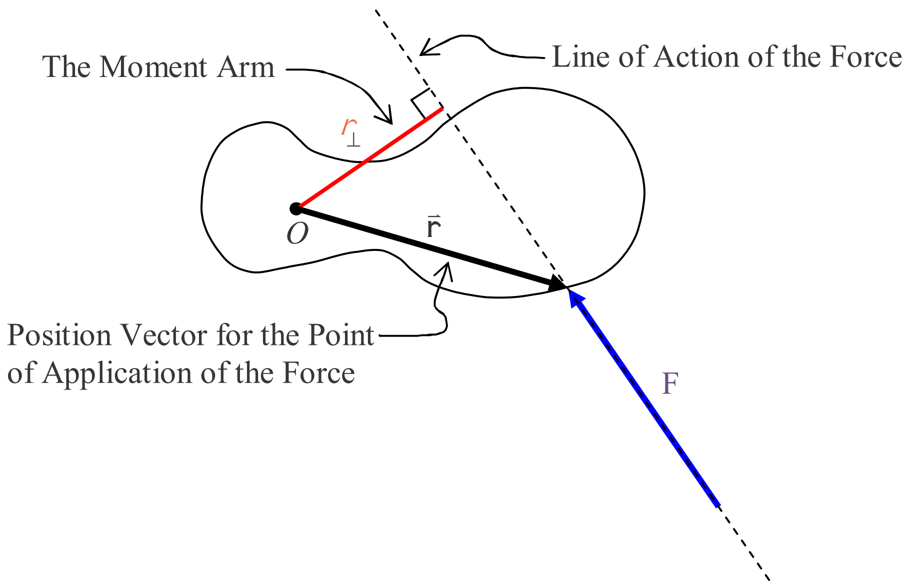
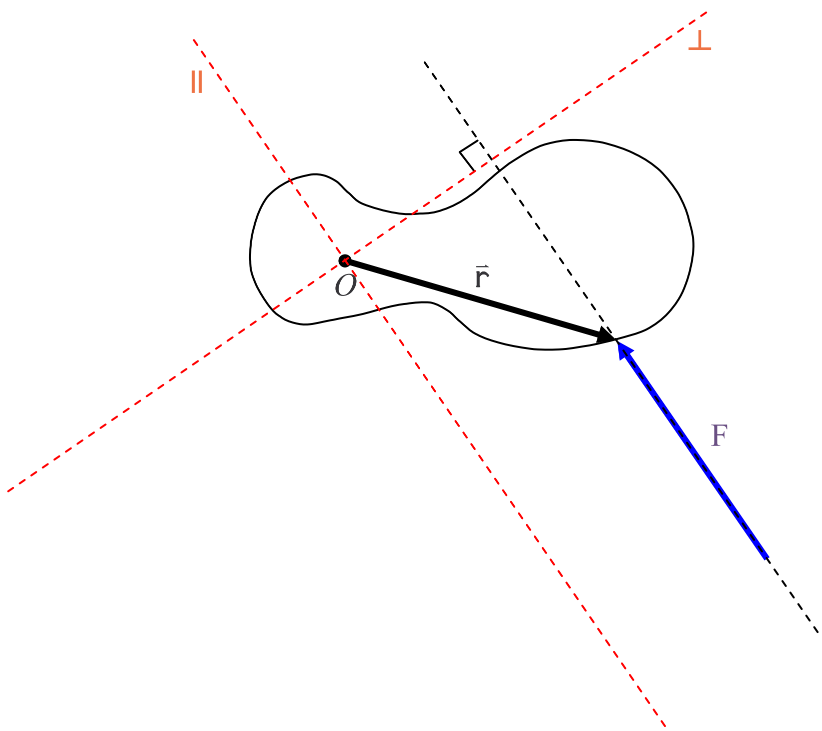
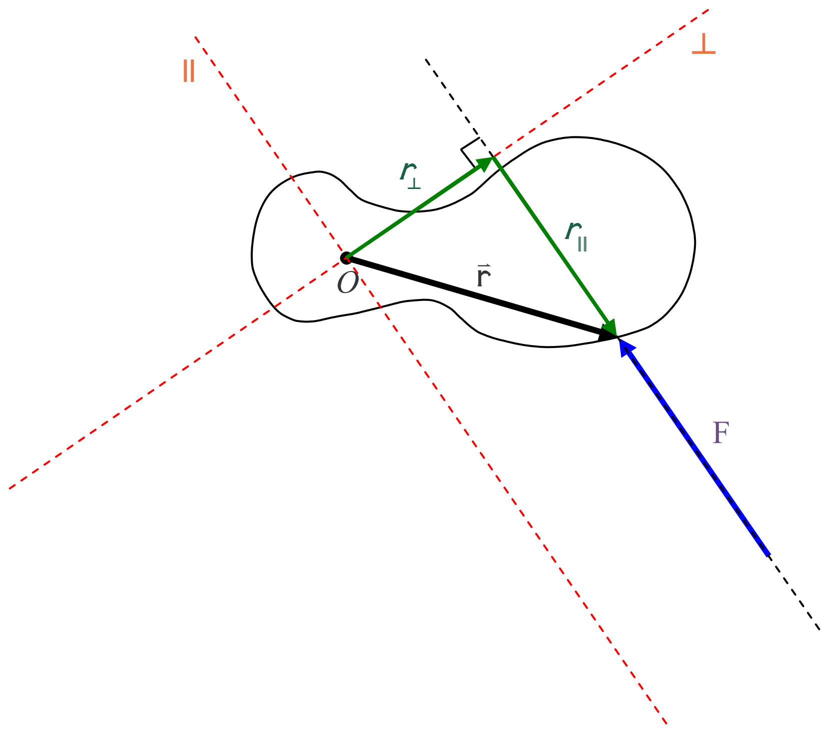
[1] You are much more familiar with relational operators then you might realize. The + sign is a relational operator for scalars (numbers). The operation is addition. Applying it to the numbers 2 and 3 yields 2+3=5. You are also familiar with the relational operators –, ·, and ÷ for subtraction, multiplication, and division (of scalars) respectively.



The moment arm can actually be defined in terms of the position vector for the point of application of the force. Consider a tilted *x*-y coordinate system, having an origin on the axis of rotation, with one axis parallel to the line of action of the force and one axis perpendicular to the line of action of the force. We label the *x* axis ⊥ for "perpendicular" and the y axis || for "parallel."



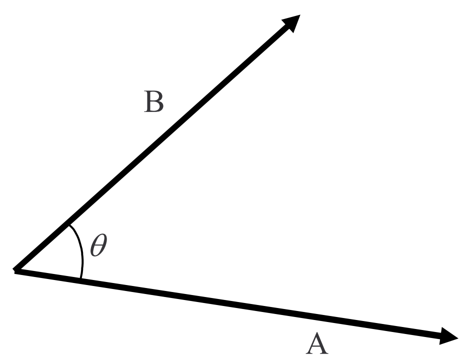
Now we break up the position vector $\vec{r}$ into its component vectors along the $\perp$ and $\parallel$ axes.



From the diagram it is clear that the moment arm $r_\perp$ is just the magnitude of the component vector, in the perpendicular-to-the-force direction, of the position vector of the point of application of the force.

Now let's discuss the cross product in general terms. Consider two vectors, $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ that are neither parallel nor **anti-parallel**[2] to each other. Two such vectors define a plane.

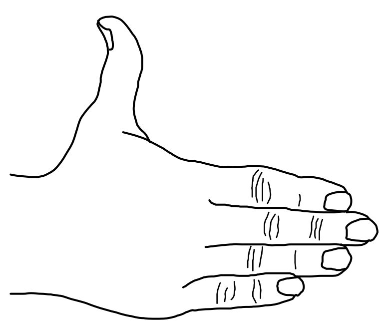
Let that plane be the plane of the page and define θ to be the smaller of the two angles between the two vectors when the vectors are drawn tail to tail.



The magnitude of the cross product vector $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ is given by

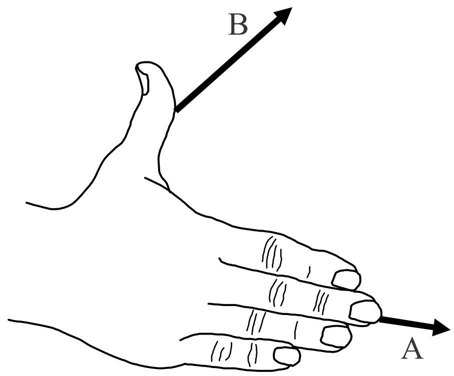
$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = AB\sin\theta \tag{21-2}$$

The direction of the cross product vector $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$ is given by the right-hand rule for the cross product of two **vectors**[3]. To apply this right-hand rule, extend the fingers of your right hand so that they are pointing directly away from your right elbow. Extend your thumb so that it is at right angles to your fingers.

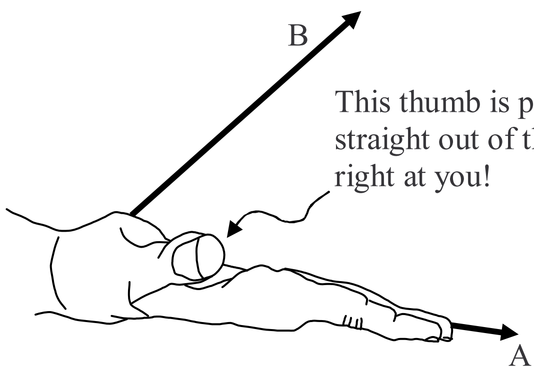


[2] Two vectors that are anti-parallel are in exact opposite directions to each other. The angle between them is 180° degrees. Anti-parallel vectors lie along parallel lines or along one and the same line, but point in opposite directions.
[3] You need to learn two right-hand rules for this course: the "right-hand rule for something curly something straight," and this one, the right-hand rule for the cross product of two vectors.

Keeping your fingers aligned with your forearm, point your fingers in the direction of the first vector (the one that appears before the "×" in the mathematical expression for the cross product; e.g. the $\vec{\mathbf{A}}$ in $\vec{\mathbf{A}}\times\vec{\mathbf{B}}$).



Now rotate your hand, as necessary, about an imaginary axis extending along your forearm and along your middle finger, until your hand is oriented such that, if you were to close your fingers, they would point in the direction of the second vector.



Your thumb is now pointing in the direction of the cross product vector. $\vec{\mathbf{C}}=\vec{\mathbf{A}}\times\vec{\mathbf{B}}$. The cross product vector $\vec{\mathbf{C}}$ is always perpendicular to both of the vectors that are in the cross product (the $\vec{\mathbf{A}}$ and the $\vec{\mathbf{B}}$ in the case at hand). Hence, if you draw them so that both of the vectors that are in the cross product are in the plane of the page, the cross product vector will always be perpendicular to the page, either straight into the page, or straight out of the page. In the case at hand, it is straight out of the page.

When we use the cross product to calculate the torque due to a force $\vec{\mathbf{F}}$ whose point of application has a position vector $\vec{\mathbf{r}}$, relative to the point about which we are calculating the torque, we get an axial torque vector $\vec{\boldsymbol{\tau}}$. To determine the sense of rotation that such a torque vector would correspond to, about the axis defined by the torque vector itself, we use The Right Hand Rule For Something Curly Something Straight. Note that we are calculating the torque with respect to a point rather than an axis—the axis about which the torque acts, comes out in the answer.

Calculating the Cross Product of Vectors that are Given in $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ Notation

Unit vectors allow for a straightforward calculation of the cross product of two vectors under even the most general circumstances, e.g. circumstances in which each of the vectors is pointing in an arbitrary direction in a three-dimensional space. To take advantage of the method, we need to know the cross product of the Cartesian coordinate axis unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ with each other.

First off, we should note that any vector crossed into itself gives zero. This is evident from equation 21-2:

$$\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB\sin\theta\,,$$

because if A and B are in the same direction, then $\theta = 0°$, and since $\sin 0° = 0$, we have $\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = 0$. Regarding the unit vectors, this means that:

$$\hat{\imath} \times \hat{\imath} = 0$$
$$\hat{\jmath} \times \hat{\jmath} = 0$$
$$\hat{k} \times \hat{k} = 0$$

Next we note that the magnitude of the cross product of two vectors that are perpendicular to each other is just the ordinary product of the magnitudes of the vectors. This is also evident from equation 21-2:

$$\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB\sin\theta\,,$$

because if $\vec{\mathbf{A}}$ is perpendicular to $\vec{\mathbf{B}}$ then $\theta = 90°$ and $\sin 90° = 1$ so

$$\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB$$

Now if $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are unit vectors, then their magnitudes are both 1, so, the product of their magnitudes is also 1. Furthermore, the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are all perpendicular to each other so the *magnitude* of the cross product of any one of them with any other one of them is the product of the two magnitudes, that is, 1.

Now how about the direction? Let's use the right hand rule to get the direction of $\hat{\imath} \times \hat{\jmath}$:

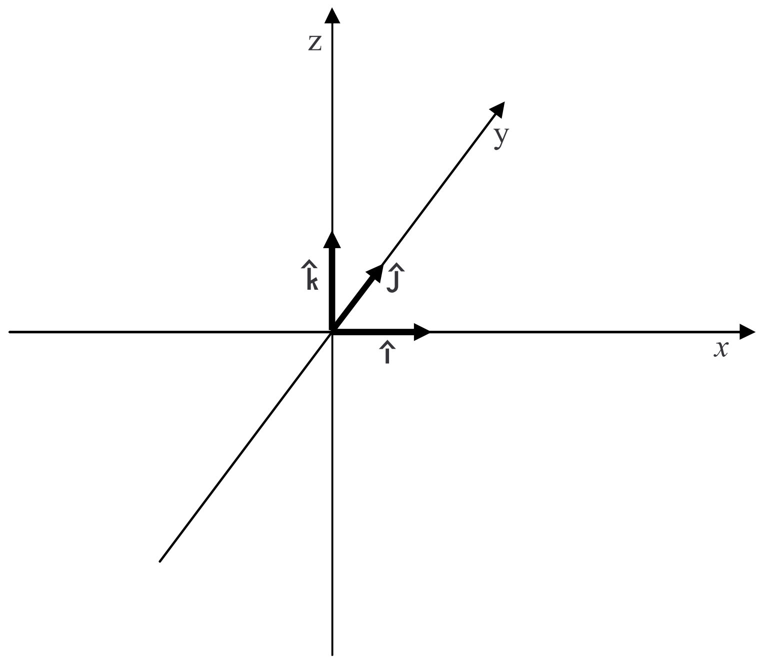


Figure 1

With the fingers of the right hand pointing directly away from the right elbow, and in the same direction as $\hat{\imath}$, (the *first* vector in "$\hat{\imath} \times \hat{\jmath}$") to make it so that if one were to close the fingers, they would point in the same direction as $\hat{\jmath}$, the palm must be facing in the +y direction. That being the case, the extended thumb must be pointing in the +z direction. Putting the magnitude (the magnitude of each unit vector is 1) and direction (+z) information together we **see**[4] that $\hat{\imath} \times \hat{\jmath} = \hat{k}$. Similarly: $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$, $\hat{\jmath} \times \hat{\imath} = -\hat{k}$, $\hat{k} \times \hat{\jmath} = -\hat{\imath}$, and $\hat{\imath} \times \hat{k} = -\hat{\jmath}$. One way of remembering this is to write $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ twice in succession:

$\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$, $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$

Then, crossing any one of the first three vectors into the vector immediately to its right yields the next vector to the right. But crossing any one of the last three vectors into the vector

[4] You may have picked up on a bit of circular reasoning here. Note that in Figure 1, if we had chosen to have the z axis point in the opposite direction (keeping *x* and y as shown) then $\hat{\imath} \times \hat{\jmath}$ would be pointing in the –z direction. In fact, having chosen the +*x* and +y directions, we define the +z direction as that direction that makes $\hat{\imath} \times \hat{\jmath} = \hat{k}$. Doing so forms what is referred to as a right-handed coordinate system which is, by convention, the kind of coordinate system that we use in science and mathematics. If $\hat{\imath} \times \hat{\jmath} = -\hat{k}$ then you are dealing with a left-handed coordinate system, something to be avoided.

immediately to its left yields the *negative* of the next vector to the left (left-to-right "+", but right-to-left "–").

Now we're ready to look at the general case. Any vector $\vec{\mathbf{A}}$ can be expressed in terms of unit vectors:

$$\vec{\mathbf{A}} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

Doing the same for a vector $\vec{\mathbf{B}}$ then allows us to write the cross product as:

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\times(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

Using the distributive rule for multiplication we can write this as:

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} = {} & A_x\hat{\imath}\times(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) + \\ & A_y\hat{\jmath}\times(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) + \\ & A_z\hat{k}\times(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})\end{aligned}$$

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} = {} & A_x\hat{\imath}\times B_x\hat{\imath} + A_x\hat{\imath}\times B_y\hat{\jmath} + A_x\hat{\imath}\times B_z\hat{k} + \\ & A_y\hat{\jmath}\times B_x\hat{\imath} + A_y\hat{\jmath}\times B_y\hat{\jmath} + A_y\hat{\jmath}\times B_z\hat{k} + \\ & A_z\hat{k}\times B_x\hat{\imath} + A_z\hat{k}\times B_y\hat{\jmath} + A_z\hat{k}\times B_z\hat{k}\end{aligned}$$

Using, in each term, the commutative rule and the associative rule for multiplication we can write this as:

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} = {} & A_xB_x(\hat{\imath}\times\hat{\imath}) + A_xB_y(\hat{\imath}\times\hat{\jmath}) + A_xB_z(\hat{\imath}\times\hat{k}) + \\ & A_yB_x(\hat{\jmath}\times\hat{\imath}) + A_yB_y(\hat{\jmath}\times\hat{\jmath}) + A_yB_z(\hat{\jmath}\times\hat{k}) + \\ & A_zB_x(\hat{k}\times\hat{\imath}) + A_zB_y(\hat{k}\times\hat{\jmath}) + A_zB_z(\hat{k}\times\hat{k})\end{aligned}$$

Now we evaluate the cross product that appears in each term:

$$\begin{aligned}\vec{\mathbf{A}}\times\vec{\mathbf{B}} = {} & A_xB_x(0) + A_xB_y(\hat{k}) + A_xB_z(-\hat{\jmath}) + \\ & A_yB_x(-\hat{k}) + A_yB_y(0) + A_yB_z(\hat{\imath}) + \\ & A_zB_x(\hat{\jmath}) + A_zB_y(-\hat{\imath}) + A_zB_z(0)\end{aligned}$$

Eliminating the zero terms and grouping the terms with $\hat{\imath}$ together, the terms with $\hat{\jmath}$ together, and the terms with $\hat{k}$ together yields:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = A_y B_z (\hat{\imath}) + A_z B_y (-\hat{\imath}) + A_z B_x (\hat{\jmath}) + A_x B_z (-\hat{\jmath}) + A_x B_y (\hat{\mathbf{k}}) + A_y B_x (-\hat{\mathbf{k}})$$

Factoring out the unit vectors yields:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{\mathbf{k}}$$

which can be written on one line as:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{\mathbf{k}} \qquad (21\text{-}3)$$

This is our end result. We can arrive at this result much more quickly if we borrow a tool from that branch of mathematics known as linear algebra (the mathematics of matrices).

We form the 3×3 matrix

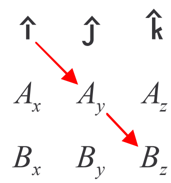
$$\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

by writing $\hat{\imath}$, $\hat{\jmath}$, $\hat{\mathbf{k}}$ as the first row, then the components of the *first* vector that appears in the cross product as the second row, and finally the components of the second vector that appears in the cross product as the last row. It turns out that the cross product is equal to the *determinant* of that matrix. We use absolute value signs on the entire matrix to signify "the determinant of the matrix." So we have:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad (21\text{-}4)$$

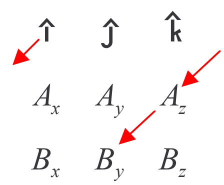
To take the determinant of a 3×3 matrix you work your way across the top row. For each element in that row you take the product of the elements along the diagonal that extends down and to the right, minus the product of the elements down and to the left; and you add the three results (one result for each element in the top row) together. If there are no elements down and to the appropriate side, you move over to the other side of the matrix (see below) to complete the diagonal.

For the first element of the first row, the $\hat{\imath}$, take the product down and to the right,



(this yields $\hat{\imath}\, A_y B_z$)

minus the product down and to the left



(the product down-and-to-the-left is $\hat{\imath}\, A_z B_y$).

For the first element in the first row, we thus have: $\hat{\imath}\, A_y B_z - \hat{\imath}\, A_z B_y$ which can be written as: $(A_y B_z - A_z B_y)\hat{\imath}$. Repeating the process for the second and third elements in the first row (the $\hat{\jmath}$ and the $\hat{k}$) we get $(A_z B_x - A_x B_z)\hat{\jmath}$ and $(A_x B_y - A_y B_x)\hat{k}$ respectively. Adding the three results, to form the determinant of the matrix results in:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k} \tag{21-3}$$

as we found before, “the hard way.”