24 Work and Energy

You have done quite a bit of problem solving using energy concepts. Back in chapter 2 we defined energy as a transferable physical quantity that an object can be said to have and we said that if one transfers energy to a material particle that is initially at rest, that particle acquires a speed which is an indicator of how much energy was transferred. We said that an object can have energy because it is moving (kinetic energy), or due to its position relative to some other object (potential energy)[1]. We said that energy has units of joules. You have dealt with translational kinetic energy $K=\frac{1}{2}mv^2$, rotational kinetic energy $K=\frac{1}{2}I\omega^2$, spring potential energy $U=\frac{1}{2}kx^2$, near-earth's-surface gravitational potential energy $U=mgy$, and the universal gravitational potential energy $U=-\frac{Gm_1m_2}{r}$ corresponding to the Universal Law of Gravitation. The principle of the conservation of energy is, in the opinion of this author, the central most important concept in physics. Indeed, at least one dictionary defines physics as the study of energy. It is important because it is conserved and the principle of conservation of energy allows us to use simple accounting procedures to make predictions about the outcomes of physical processes that have yet to occur and to understand processes that have already occurred. According to the principle of conservation of energy, any change in the total amount of energy of a system can be accounted for in terms of energy transferred from the immediate surroundings to the system or to the immediate surroundings from the system. Physicists recognize two categories of energy transfer processes. One is called work and the other is called heat flow. In this chapter we focus our attention on work.

Conceptually, positive work is what you are doing on an object when you push or pull on it in the same direction in which the object is moving. You do negative work on an object when you push or pull on it in the direction opposite the direction in which the object is going. The mnemonic for remembering the definition of work that helps you remember how to calculate it is "Work is Force times Distance." The mnemonic does not tell the whole story. It is good for the case of a constant force acting on an object that moves on a straight line path when the force is in the same exact direction as the direction of motion.

A more general, but still not completely general, "how-to-calculate-it" definition of work applies to the case of a constant force acting on an object that moves along a straight line path (when the force is not necessarily directed along the path). In such a case, the work W done on the object, when it travels a certain distance along the path, is: the along-the-path component of the force $F_{\parallel}$ times the length of the path segment Δr.

$$W=F_{\parallel}\Delta r \tag{24-1}$$

Even this case still needs some additional clarification: If the force component vector along the path is in the same direction as the object's displacement vector, then $F_{\parallel}$ is positive, so the work is positive; but if the force component vector along the path is in the opposite direction to that of

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[1] As mentioned before, the potential energy is actually the energy of the system of the objects and their fields as a whole, but it is common to assign it to part of the system for "bookkeeping" purposes as I do in this book.

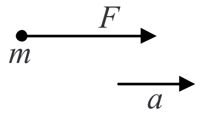
the object's displacement vector, then $F_{||}$ is negative, so the work is negative. Thus, if you are pushing or pulling on an object in a direction that would tend to make it speed up, you are doing positive work on the object. But if you are pushing or pulling on an object in a direction that would tend to slow it down, you are doing negative work on the object.

In the most general case in which the "component of the force along the path" is continually changing because the force is continually changing (such as in the case of an object on the end of a spring) or because the path is not straight, our "how-to-calculate-it" definition of the work becomes: For each infinitesimal path segment making up the path in question, we take the product of the along-the-path force component and the infinitesimal length of the path segment. The work is the sum of all such products. Such a sum would have an infinite number of terms. We refer to such a sum as an integral.

The Relation Between Work and Motion

Let's go back to the simplest case, the case in which a force $\vec{\mathbf{F}}$ is the only force acting on a particle of mass m which moves a distance Δr (while the force is acting on it) in a straight line in the exact same direction as the force. The plan here is to investigate the connection between the work on the particle and the motion of the particle. We'll start with Newton's 2nd Law.

Free Body Diagram



$$a_{\rightarrow} = \frac{1}{m}\sum F_{\rightarrow}$$

$$a = \frac{1}{m}F$$

Solving for F, we arrive at:

$$F = ma$$

On the left, we have the magnitude of the force. If we multiply that by the distance Δr, we get the work done by the force on the particle as it moves the distance Δr along the path, in the same direction as the force. If we multiply the left side of the equation by Δr then we have to multiply the right by the same thing to maintain the equality.

$$F\Delta r = ma\Delta r$$

On the left we have the work W, so:

$$W = m a \Delta r$$

On the right we have two quantities used to characterize the motion of a particle so we have certainly met our goal of relating work to motion, but we can untangle things on the right a bit if we recognize that, since we have a constant force, we must have a constant acceleration. This means the constant acceleration equations apply, in particular, the one that (in terms of r rather than x) reads:

$$v^2 = v_o^2 + 2a\Delta r$$

Solving this for $a\Delta r$ gives

$$a\Delta r = \frac{1}{2}v^2 - \frac{1}{2}v_o^2$$

Substituting this into our expression for W above (the one that reads $W = m a \Delta r$) we obtain

$$W = m\left(\frac{1}{2}v^2 - \frac{1}{2}v_o^2\right)$$

which can be written as

$$W = \frac{1}{2}m v^2 - \frac{1}{2}m v_o^2$$

Of course we recognize the $\frac{1}{2}m v_o^2$ as the kinetic energy of the particle before the work is done on the particle and the $\frac{1}{2}m v^2$ as the kinetic energy of the particle after the work is done on it. To be consistent with the notation we used in our early discussion of the conservation of mechanical energy we change to the notation in which the prime symbol ($'$) signifies "after" and no super- or subscript at all (rather than the subscript "o") represents "before." Using this notation and the definition of kinetic energy, our expression for W becomes:

$$W = K' - K$$

Since the "after" kinetic energy minus the "before" kinetic energy is just the change in kinetic energy ΔK, we can write the expression for W as:

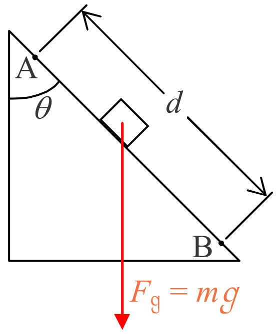
$$W = \Delta K \tag{24-2}$$

This is indeed a simple relation between work and motion. The cause, work on a particle, on the left, is exactly equal to the effect, a change in the kinetic energy of the particle. This result is so important that we give it a name, it is the *Work-Energy Relation*. It also goes by the name: *The Work-Energy Principle*. It works for extended rigid bodies as well. In the case of a rigid body that rotates, it is the displacement of the point of application of the force, along the path of said point of application, that is used (as the Δr) in calculating the work done on the object.

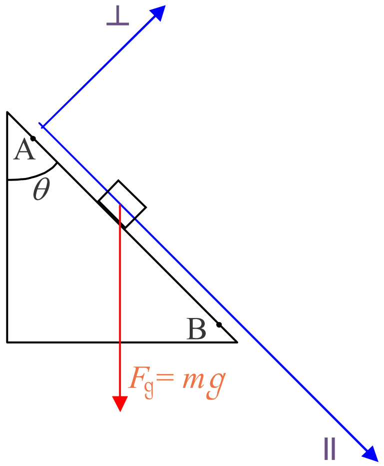
In the expression $W = \Delta K$, the work is the net work (the total work) done by all the forces acting on the particle or rigid body. The net work can be calculated by finding the work done by each force and adding the results, or by finding the net force and using it in the definition of the work.

Calculating the Work as the Force-Along-the-Path Times the Length of the Path

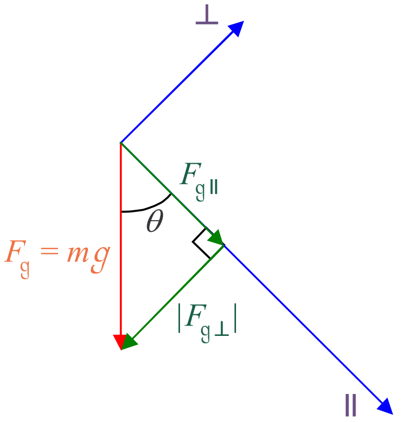
Consider a block on a flat frictionless incline that makes an angle θ with the vertical. The block travels from a point A near the top of the incline to a point B, a distance d in the down-the-incline direction from A. Find the work done, by the gravitational force, on the block.



We've drawn a sketch of the situation (not a free body diagram). We note that the force for which we are supposed to calculate the work is not along the path. So, we define a coordinate system with one axis in the down-the-incline direction and the other perpendicular to that axis

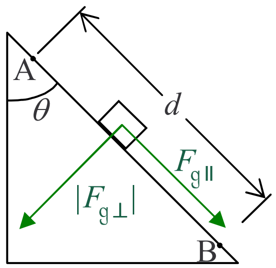


and break the gravitational force vector up into its components with respect to that coordinate system.



$$F_{g\parallel} = F_g \cos\theta = mg\cos\theta$$
$$\left|F_{g\perp}\right| = F_g \sin\theta = mg\sin\theta$$

Now we redraw the sketch with the gravitational force replaced by its components:



$F_{g\perp}$, being perpendicular to the path does no work on the block as the block moves from A to B. The work done by the gravitational force is given by

$$W = F_{\parallel}\, d$$

$$W = F_{g_{\parallel}}\, d$$

$$W = mg(\cos\theta)\, d$$

$$W = mgd\cos\theta$$

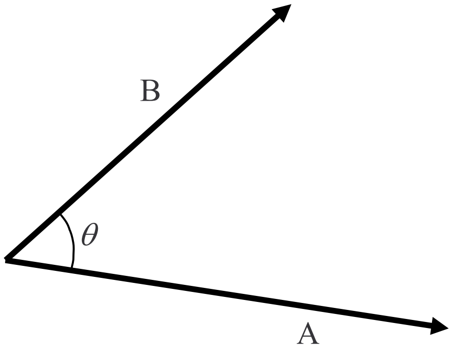
While this method for calculating the work done by a force is perfectly valid, there is an easier way. It involves another product operator for vectors (besides the cross product), called the *dot product*. To use it, we need to recognize that the length of the path, combined with the direction of motion, is none other than the displacement vector (for the point of application of the force). Then we just need to find the dot product of the force vector and the displacement vector.

The Dot Product of Two Vectors

The dot product of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is written $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ and is expressed as:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\theta \tag{24-3}$$

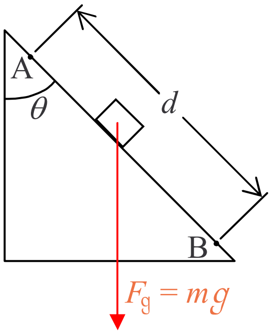
where θ, just as in the case of the cross product, is the angle between the two vectors after they have been placed tail to tail.



The dot product can be interpreted as either $A_{||}B$ (the component of $\vec{\mathbf{A}}$ along $\vec{\mathbf{B}}$, times, the magnitude of $\vec{\mathbf{B}}$) or $B_{||}A$ (the component of $\vec{\mathbf{B}}$ along $\vec{\mathbf{A}}$, times, the magnitude of $\vec{\mathbf{A}}$), both of which evaluate to one and the same value. This makes the dot product perfect for calculating the work. Since $\vec{\mathbf{F}} \cdot \overrightarrow{\Delta\mathbf{r}} = F_{||}\Delta r$ and $F_{||}\Delta r$ is W, we have

$$W = \vec{\mathbf{F}} \cdot \overrightarrow{\Delta\mathbf{r}} \tag{24-4}$$

By means of the dot product, we can solve the example in the last section much more quickly than we did before.



Find the work done on the block by the gravitational force when the object moves from point A to Point B.

We define the displacement vector $\vec{\mathbf{d}}$ to have a magnitude equal to the distance from point A to point B with a direction the same as the direction of motion (the down-the-ramp direction).

Using our definition of work as the dot product of the force and the displacement, equation 24-4:

$$W = \vec{\mathbf{F}} \cdot \overrightarrow{\Delta \mathbf{r}}$$

with the gravitational force vector $\vec{\mathbf{F}}_g$ being the force, and $\vec{\mathbf{d}}$ being the displacement, the work can be written as:

$$W = \vec{\mathbf{F}}_g \cdot \vec{\mathbf{d}}\,.$$

Using the definition of the dot product we find that:

$$W = F_g d \cos\theta\,.$$

Replacing the magnitude of the gravitational force with mg we arrive at our final answer:

$$W = mgd\cos\theta\,.$$

This is the same answer that we got prior to our discussion of the dot product.

In cases in which the force and the displacement vectors are given in $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ notation, finding the work is straightforward.

The Dot Product in Unit Vector Notation

The simple dot product relations among the unit vectors makes it easy to evaluate the dot product of two vectors expressed in unit vector notation. From what amounts to our definition of the dot product, equation 24-3:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\theta$$

we note that a vector dotted into itself is simply the square of the magnitude of the vector. This is true because the angle between a vector and itself is 0° and $\cos 0°$ is 1.

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{A}} = AA\cos 0^\circ = A^2$$

Since the unit vectors all have magnitude 1, any unit vector dotted into itself yields $(1)^2$ which is just 1.

$$\hat{\imath} \cdot \hat{\imath} = 1, \quad \hat{\jmath} \cdot \hat{\jmath} = 1, \quad \text{and} \quad \hat{k} \cdot \hat{k} = 1$$

Now the angle between any two different Cartesian coordinate axis unit vectors is 90° and the $\cos 90°$ is 0. Thus, the dot product of any Cartesian coordinate axis unit vector into any other Cartesian coordinate axis unit vector is zero.

So, if

$$\vec{\mathbf{A}} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{\mathbf{k}}$$

and

$$\vec{\mathbf{B}} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{\mathbf{k}}$$

then $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ is just

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{\mathbf{k}}) \cdot (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{\mathbf{k}})$$

$$\begin{aligned}\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = {} & A_x\hat{\imath} \cdot (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{\mathbf{k}}) + \\ & A_y\hat{\jmath} \cdot (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{\mathbf{k}}) + \\ & A_z\hat{\mathbf{k}} \cdot (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{\mathbf{k}})\end{aligned}$$

$$\begin{aligned}\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = {} & A_x\hat{\imath} \cdot B_x\hat{\imath} + A_x\hat{\imath} \cdot B_y\hat{\jmath} + A_x\hat{\imath} \cdot B_z\hat{\mathbf{k}} + \\ & A_y\hat{\jmath} \cdot B_x\hat{\imath} + A_y\hat{\jmath} \cdot B_y\hat{\jmath} + A_y\hat{\jmath} \cdot B_z\hat{\mathbf{k}} + \\ & A_z\hat{\mathbf{k}} \cdot B_x\hat{\imath} + A_z\hat{\mathbf{k}} \cdot B_y\hat{\jmath} + A_z\hat{\mathbf{k}} \cdot B_z\hat{\mathbf{k}}\end{aligned}$$

$$\begin{aligned}\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = {} & A_xB_x\hat{\imath} \cdot \hat{\imath} + A_xB_y\hat{\imath} \cdot \hat{\jmath} + A_xB_z\hat{\imath} \cdot \hat{\mathbf{k}} + \\ & A_yB_x\hat{\jmath} \cdot \hat{\imath} + A_yB_y\hat{\jmath} \cdot \hat{\jmath} + A_yB_z\hat{\jmath} \cdot \hat{\mathbf{k}} + \\ & A_zB_x\hat{\mathbf{k}} \cdot \hat{\imath} + A_zB_y\hat{\mathbf{k}} \cdot \hat{\jmath} + A_zB_z\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}\end{aligned}$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$$

The end result is that the dot product of two vectors is simply the sum of: the product of the two vectors' x components, the product of their y components, and the product of their z components.

Energy Transfer Work vs. Center of Mass Pseudo-Work

I introduced the topic of work by stating that it represents one category of energy transfer to a system. As such, work is energy transfer work. There is a quantity that is calculated in much the same way as work, with one subtle difference. I'm going to call the quantity *center of mass pseudo-work* and I'm going to use a couple of specific processes involving a frictionless horizontal surface, a spring, and a block (and in one case, another block) to distinguish energy transfer work from center of mass pseudo-work. Suppose we attach the spring to the wall so that the spring sticks out horizontally and then push the block toward the wall in such a manner as to compress the spring. Then we release the block from rest and start our observations at the first instant subsequent to release. Let our system be the block. The spring pushes the block away from the wall. The spring transfers energy to the block while the spring is in contact with the block. The work done can be calculated as the integral of vector force dot vector infinitesimal displacement which I'll loosely state as the integral of force times distance. The distance in this case is the displacement (the infinite set of infinitesimal displacements) of the point of application of the force. This kind of work is energy transfer work. It is the amount of energy transferred to the block by the spring.

Now let's disconnect the spring from the wall and attach the spring to the block so that the spring sticks out horizontally from the block and again push the block up against the wall, compressing the spring, and release the block from rest. Let our system be the block plus spring. The block goes sliding off as before, this time with the spring attached. The wall does no energy-transfer work on the system because the part of the wall that is exerting the force on the system is not moving—there is no displacement. However, we find something useful if we calculate the integral of the vector force (exerted by the wall) dot vector infinitesimal displacement of the center of mass of the system—loosely stated, force times distance moved by center of mass. I'm calling that "something useful" the center of mass pseudo-work experienced by the system. It's useful because our Newton's Law derivation shows that quantity to be equal to the change in the center of mass kinetic energy of the system. In this case the system gained some center of mass kinetic energy even though no energy was transferred to it. How did that happen? Energy that was already part of the system, energy stored in the spring, was converted to center of mass kinetic energy.

So what is the subtle difference? In both cases we are, loosely speaking, calculating force times distance. But in the case of energy transfer work, the distance is the distance moved by that element of the agent of the force that is in contact with the victim at exactly that point where the force is being applied, whereas, in the case of center of mass work, the distance in "force times distance" is the distance moved by the center of mass. For a particle, there is no difference. For a truly rigid body undergoing purely translational motion (no rotation) there is no difference. But beware, a truly rigid body is an idealized object in which no bit of the body can move relative to any other bit of the body. Even for such a body, if there is rotation, there will be a difference between the energy transfer work and the center of mass pseudo-work done on the object. Consider for instance a block at rest on a horizontal frictionless surface. You apply an off-center horizontal force to the block for a short distance by pressing on the block with your finger. The work you do is the integrated force times distance over which you move the tip of your finger. It will be greater than the integrated force times the distance over which the center of mass moves. Some of the work you do goes into increasing the center of mass kinetic energy of the rigid body and some of it goes into increasing the rotational kinetic energy of the rigid body. In this case the energy transfer work is greater than the center of mass pseudo-work.

Concluding Remarks

At this point you have two ways of calculating the work done on an object. If you are given information about the force and the path you will use the "force times distance" definition of work. But if you are given information on the effect of the work (the change in kinetic energy) then you will determine the value of the change in kinetic energy and substitute that into the work energy relation, equation 24-2:

$$W = \Delta K$$

to determine the work (or center of mass pseudo-work as applicable). There is yet another method for calculating the work. Like the first method, it is good for cases in which you have information on the force and the path. It only works for certain kinds of forces, but when it does work, to use it, the only thing you need to know about the path is the positions of the endpoints. This third method for calculating the work involves the *potential energy*, the main topic of our next chapter.