# 25 Potential Energy, Conservation of Energy, Power

The work done on a particle by a force acting on it as that particle moves from point A to point B under the influence of that force, for *some* forces, does not depend on the path followed by the particle. For such a force there is an easy way to calculate the work done on the particle as it moves from point A to point B. One simply has to assign a value of potential energy (of the particle[1]) to point A (call that value $U_A$) and a value of potential energy to point B (call that value $U_B$). One chooses the values such that the work done by the force in question is just the negative of the difference between the two values.

$$W = -(U_B - U_A)$$

$$W = -\Delta U \tag{25-1}$$

$\Delta U = U_B - U_A$ is the change in the potential energy experienced by the particle as it moves from point A to point B. The minus sign in equation 25-1 ensures that an increase in potential energy corresponds to negative work done by the corresponding force. For instance for the case of near-earth's-surface gravitational potential energy, the associated force is the gravitational force, a.k.a. the gravitational force. If we lift an object upward near the surface of the earth, the gravitational force does negative work on the object since the (downward) force is in the opposite direction to the (upward) displacement. At the same, time, we are increasing the capacity of the particle to do work so we are increasing the potential energy. Thus, we need the "–" sign in $W = -\Delta U$ to ensure that the change in potential energy method of calculating the work gives the same algebraic sign for the value of the work that the force-along-the path times the length of the path gives.

Note that in order for this method of calculating the work to be useful in any case that might arise, one must assign a value of potential energy to every point in space where the force can act on a particle so that the method can be used to calculate the work done on a particle as the particle moves from any point A to any point B. In general, this means we need a value for each of an infinite set of points in space.

This assignment of a value of potential energy to each of an infinite set of points in space might seem daunting until you realize that it can be done by means of a simple algebraic expression. For instance, we have already written the assignment for a particle of mass $m_2$ for the case of the universal gravitational force due to a particle of mass $m_1$. It was equation 17-5:

$$U = -\frac{G m_1 m_2}{r}$$

[1] The potential energy is actually the potential energy of the system consisting of the particle, whatever the particle is interacting with, and the relevant field. For instance, if we are talking about a particle in the gravitational field of the earth, the potential energy under discussion is the potential energy of the earth plus particle and gravitational field of the earth plus particle. For accounting purposes, it is convenient to ascribe the potential energy to the particle and that is what I do in this book.

in which $G$ is the universal gravitational constant $G = 6.67 \times 10^{-11} \dfrac{\text{N} \cdot \text{m}^2}{\text{kg}^2}$ and $r$ is the distance that particle 2 is from particle 1. Note that considering particle 1 to be at the origin of a coordinate system, this equation assigns a value of potential energy to every point in the universe! The value, for any point, simply depends on the distance that the point is from the origin. Suppose we want to find the work done by the gravitational force due to particle 1, on particle 2 as particle 2 moves from point A, a distance $r_A$ from particle 1 to point B, a distance $r_B$ from particle 1. The gravitational force exerted on it (particle 2) by the gravitational field of particle 1 does an amount of work, on particle 2, given by (starting with equation 25-1):

$$W = -\Delta U$$

$$W = -(U_B - U_A)$$

$$W = -\left[\left(-\frac{G m_1 m_2}{r_B}\right) - \left(-\frac{G m_1 m_2}{r_A}\right)\right]$$

$$W = G m_1 m_2 \left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

## *The Relation Between a Conservative Force and the Corresponding Potential*

While this business of calculating the work done on a particle as the negative of the change in its potential energy does make it a lot easier to calculate the work, we do have to be careful to define the potential such that this method is equivalent to calculating the work as the force-along-the-path times the length of the path.

Rather than jump into the problem of finding the potential energy at all points in a three-dimensional region of space for a kind of force known to exist at all points in that three-dimensional region of space, let's look into the simpler problem of finding the potential along a line. We define a coordinate system consisting of a single axis, let's call it the $x$-axis, with an origin and a positive direction. We put a particle on the line, a particle that can move along the line. We assume that we have a force that acts on the particle wherever the particle is on the line and that the force is directed along the line. While we will also address the case of a force which has the same value at different points along the line, we assume that, in general, *the force varies with position*. ←Remember this fact so that you can find the flaw discussed below. Because we want to define a potential for it, it is important that the work done on the particle by the force being exerted on the particle, as the particle moves from point A to point B does not depend on how the particle gets from point A to point B. Our goal is to define a potential energy function for the force such that we get the same value for the work done on the particle by the force whether we use the force-along-the-path method to calculate it or the negative of the change of potential energy method. Suppose the particle undergoes a displacement $\Delta x$ along the line under the influence of the force. See if you can see the flaw in the following, before I point it out: We write $W = F\,\Delta x$ for the work done by the force, calculated using the force-along-the-path times the length of the path idea, and then $W = -\Delta U$ for the work done by the force calculated using the negative of the change in potential energy concept. Setting the two expressions equal to each other, we have, $F\Delta x = -\Delta U$ which we can write as $F = -\dfrac{\Delta U}{\Delta x}$ for the relation between the potential energy and the $x$-component of the force.

Do you see where we went wrong? While the method will work for the special case in which the force is a constant, we were supposed to come up with a relation that was good for the general case in which the force varies with position. That means that for each value of $x$ in the range of values extending from the initial value, let's call it $x_A$, to the value at the end of the displacement $x_A + \Delta x$, there is a different value of force. So the expression $W = F\,\Delta x$ is inappropriate. Given a numerical problem, there is no one value to plug in for $F$, because $F$ varies along the $\Delta x$.

To fix things, we can shrink $\Delta x$ to infinitesimal size, so small that, $x_A$ and $x_A + \Delta x$ are, for all practical purposes, one and the same point. That is to say, we take the limit as $\Delta x \to 0$. Then our relation becomes

$$F_x = \lim_{\Delta x \to 0}\left(-\frac{\Delta U}{\Delta x}\right)$$

which is the same thing as

$$F_x = -\lim_{\Delta x \to 0}\left(\frac{\Delta U}{\Delta x}\right)$$

The limit of $\dfrac{\Delta U}{\Delta x}$ that appears on the right is none other than the derivative $\dfrac{dU}{dx}$, so:

$$F_x = -\frac{dU}{dx} \tag{25-2}$$

To emphasize the fact that force is a vector, we write it in unit vector notation as:

$$\vec{\mathbf{F}} = -\frac{dU}{dx}\,\hat{\imath} \tag{25-3}$$

Let's make this more concrete by using it to determine the potential energy due to a force with which you are familiar—the force due to a spring.

Consider a block on frictionless horizontal surface. The block is attached to one end of a spring. The other end of the spring is attached to a wall. The spring extends horizontally away from the wall, at right angles to the wall. Define an $x$-axis with the origin at the equilibrium position of that end of the spring which is attached to the block. Consider the away-from-the-wall direction to be the positive $x$ direction. Experimentally, we find that the force exerted by the spring on the block is given by:

$$\vec{\mathbf{F}} = -k\,x\,\hat{\imath} \tag{25-4}$$

where $k$ is the force constant of the spring. (Note: A positive $x$, corresponding to the block having been pulled away from the wall, thus stretching the spring, results in a force in the negative $x$ direction. A negative $x$, compressed spring, results in a force in the $+x$ direction, consistent with common sense.) By comparison with equation 25-3 (the one that reads $\vec{\mathbf{F}} = -\frac{dU}{dx}\hat{\imath}$) we note that the potential energy function has to be defined so that

$$\frac{dU}{dx} = k\,x$$

This is such a simple case that we can pretty much guess what $U$ has to be. $U$ has to be defined such that when we take the derivative of it we get a constant (the $k$) times $x$ to the power of 1. Now when you take the derivative of $x$ to a power, you reduce the power by one. For that to result in a power of 1, the original power must be 2. Also, the derivative of a constant times something yields that same constant times the derivative, so, there must be a factor of $k$ in the potential energy function. Let's try $U = kx^2$ and see where that gets us. The derivative of $kx^2$ is $2kx$. Except for that factor of 2 out front, that is exactly what we want. Let's amend our guess by multiplying it by a factor of $\frac{1}{2}$, to eventually cancel out the 2 that comes down when we take the derivative. With $U = \frac{1}{2}k\,x^2$ we get $\frac{dU}{dx} = k\,x$ which is exactly what we needed. Thus

$$U = \frac{1}{2}k\,x^2 \tag{25-5}$$

is indeed the potential energy for the force due to a spring. You used this expression back in chapter 2. Now you know where it comes from.

We have considered two other conservative forces. For each, let's find the potential energy function $U$ that meets the criterion that we have written as, $\vec{\mathbf{F}} = -\frac{dU}{dx}\hat{\imath}$.

First, let's consider the near-earth's-surface gravitational force exerted on an object of mass $m$, by the earth. We choose our single axis to be directed vertically upward with the origin at an arbitrary but clearly specified and fixed *elevation* for the entire problem that one might solve using the concepts under consideration here. By convention, we call such an axis the $y$ axis rather than the $x$ axis. Now we know that the gravitational force is given simply (again, this is an experimental result) by

$$\vec{\mathbf{F}} = -mg\,\hat{\mathbf{j}}$$

where the $mg$ is the known magnitude of the gravitational force and the $-\hat{\mathbf{j}}$ is the downward direction.

Equation 25-3, written for the case at hand is:

$$\vec{\mathbf{F}} = -\frac{dU}{dy}\hat{\mathbf{j}}$$

For the last two equations to be consistent with each other, we need $U$ to be defined such that

$$\frac{dU}{dy} = mg$$

For the derivative of $U$ with respect to $y$ to be the constant "$mg$", $U$ must be given by

$$U = mgy \qquad (25\text{-}6)$$

and indeed this is the equation for the earth's near-surface gravitational potential energy. Please verify that when you take the derivative of it with respect to $y$, you do indeed get the magnitude of the gravitational force, $mg$.

Now let's turn our attention to the Universal Law of Gravitation. Particle number 1 of mass $m_1$ creates a gravitational field in the region of space around it. Let's define the position of particle number 1 to be the origin of a three-dimensional Cartesian coordinate system. Now let's assume that particle number 2 is at some position in space, a distance $r$ away from particle 1. Let's define the direction that particle 2 is in, relative to particle 1, as the $+x$ direction. Then, the coordinates of particle 2 are $(r, 0, 0)$. $r$ is then the $x$ component of the position vector for particle 2, a quantity that we shall now call $x$. That is, $x$ is defined such that $x = r$. In terms of the coordinate system thus defined, the force exerted by the gravitational field of particle 1, on particle 2, is given by:

$$\vec{\mathbf{F}} = -\frac{Gm_1m_2}{x^2}\hat{\mathbf{i}}$$

which I rewrite here:

$$\vec{\mathbf{F}} = -\frac{G m_1 m_2}{x^2}\,\hat{\mathbf{i}}$$

Compare this with equation 25-3:

$$\vec{\mathbf{F}} = -\frac{dU}{dx}\,\hat{\mathbf{i}}$$

Combining the two equations, we note that our expression for the potential energy $U$ in terms of $x$ must satisfy the equation

$$\frac{dU}{dx} = \frac{G m_1 m_2}{x^2}$$

It's easier to deduce what $U$ must be if we write this as

$$\frac{dU}{dx} = G m_1 m_2 x^{-2}$$

For the derivative of $U$ with respect to $x$ to be a constant ($Gm_1 m_2$) times a power ($-2$) of $x$, $U$ itself must be that same constant ($Gm_1 m_2$) times $x$ to the next higher power ($-1$), divided by the value of the latter power.

$$U = \frac{G m_1 m_2 x^{-1}}{-1}$$

which can be written

$$U = -\frac{G m_1 m_2}{x}$$

Recalling that the $x$ in the denominator is simply the distance from particle 1 to particle 2 which we have also defined to be $r$, we can write this in the form in which it is more commonly written:

$$U = -\frac{G m_1 m_2}{r} \qquad (25\text{-}7)$$

This is indeed the expression for the gravitational potential that we gave you (without any justification for it) back in Chapter 17, the chapter on the Universal Law of Gravitation.

## Conservation of Energy Revisited

Recall the work-energy relation, equation 24-2 from last chapter,

$$W = \Delta K \ ,$$

the statement that work causes a change in kinetic energy. Now consider a case in which all the work is done by conservative forces, so, the work can be expressed as the negative of the change in potential energy.

$$-\Delta U = \Delta K$$

Further suppose that we are dealing with a situation in which a particle moves from point A to point B under the influence of the force or forces corresponding to the potential energy *U*.

Then, the preceding expression can be written as:

$$-(U_{\mathrm{B}} - U_{\mathrm{A}}) = K_{\mathrm{B}} - K_{\mathrm{A}}$$

$$-U_{\mathrm{B}} + U_{\mathrm{A}} = K_{\mathrm{B}} - K_{\mathrm{A}}$$

$$K_{\mathrm{A}} + U_{\mathrm{A}} = K_{\mathrm{B}} + U_{\mathrm{B}}$$

Switching over to notation in which we use primed variables to characterize the particle when it is at point B and unprimed variables at A we have:

$$K + U = K' + U'$$

Interpreting $E = K + U$ as the energy of the system at the “before” instant, and $E' = K' + U'$ as the energy of the system at the “after” instant, we see that we have derived the conservation of mechanical energy statement for the special case of no net energy transfer to or from the surroundings and no conversion of energy within the system from mechanical energy to other forms or vice versa. In equation form, the statement is

$$E = E' \qquad (25\text{-}8)$$

an equation to which you were introduced in chapter 2. Note that you would be well advised to review chapter 2 now, because for the current chapter, you are again responsible for solving any of the “chapter-2-type” problems (remembering to include, and correctly use, before and after diagrams) and answer any of the “chapter-2-type” questions.

## Power

In this last section on energy we address a new topic. As a separate and important concept, it would deserve its own chapter except for the fact that it is such a simple, straightforward concept. *Power* is the rate of energy transfer, energy conversion, and in some cases, the rate at which transfer and conversion of energy are occurring simultaneously. When you do work on an object, you are transferring energy to that object. Suppose for instance that you are pushing a block across a horizontal frictionless surface. You are doing work on the object. The kinetic energy of the object is increasing. The rate at which the kinetic energy is increasing is referred to as power. The rate of change of any quantity (how fast that quantity is changing) can be calculated as the derivative of that quantity with respect to time. In the case at hand, the power $P$ can be expressed as

$$P = \frac{dK}{dt} \tag{25-9}$$

the time derivative of the kinetic energy. Since $K = \frac{1}{2} m v^2$ we have

$$P = \frac{d}{dt}\frac{1}{2} m v^2$$

$$P = \frac{1}{2} m \frac{d}{dt} v^2$$

$$P = \frac{1}{2} m 2 v \frac{dv}{dt}$$

$$P = m \frac{dv}{dt} v$$

$$P = m a_{\parallel} v$$

$$P = F_{\parallel} v$$

$$P = \vec{\mathbf{F}} \cdot \vec{\boldsymbol{v}} \tag{25-10}$$

where $a_{\parallel}$ is the acceleration component parallel to the velocity vector. The perpendicular component changes the direction of the velocity but not the magnitude.

Besides the rate at which the kinetic energy is changing, the power is the rate at which work is being done on the object. In an infinitesimal time interval $dt$, you do an infinitesimal amount of work

$$dW = \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dx}}$$

on the object. Dividing both sides by $dt$, we have

$$\frac{dW}{dt} = \vec{\mathbf{F}} \cdot \frac{\overrightarrow{\mathbf{dx}}}{dt}$$

which again is

$$P = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$$

as it must be since, in accord with the work-energy relation, the rate at which you do work on the object has to be the rate at which the kinetic energy of the object increases.

If you do work at a steady rate for a finite time interval, the power is constant and can simply be calculated as the amount of work done during the time interval divided by the time interval itself. For instance, when you climb stairs, you convert chemical energy stored in your body to gravitational potential energy. The rate at which you do this is power. If you climb at a steady rate for a total increase of gravitational potential energy of $\Delta U$ over a time interval $\Delta t$ then the constant value of your power during that time interval is

$$P = \frac{\Delta U}{\Delta t} \qquad (25\text{-}11)$$

If you know that the power is constant, you know the value of the power $P$, and you are asked to find the total amount of work done, the total amount of energy transferred, and/or the total amount of energy converted during a particular time interval $\Delta t$, you just have to multiply the power $P$ by the time interval $\Delta t$.

$$\text{Energy} = P\,\Delta t \qquad (25\text{-}12)$$

One could include at least a dozen formulas on your formula sheet for power, but they are all so simple that, if you understand what power is, you can come up with the specific formula you need for the case on which you are working. We include but one formula on the formula sheet,

$$P = \frac{dE}{dt} \qquad (25\text{-}13)$$

which should remind you what power is. Since power is the rate of change of energy, the SI units of power must be $\frac{\text{J}}{\text{s}}$. This combination unit is given a name, the watt, abbreviated W.

$$1\ \text{W} \equiv 1\ \frac{\text{J}}{\text{s}}$$