# 26 Impulse and Momentum

## *First, a Few More Words on Work and Energy, for Comparison Purposes*

Imagine a gigantic air hockey table with a whole bunch of pucks of various masses, none of which experiences any friction with the horizontal surface of the table. Assume air resistance to be negligible. Now suppose that you come up and give each puck a shove, where the kind of shove that you give the first one is special in that the whole time you are pushing on that puck, the force has one and the same value; and the shove that you give each of the other pucks is similar in the following respect: To each puck you apply the same force that you applied to the first puck, over the same exact distance. Since you give each of the pucks a similar shove, you might expect the motion of the pucks (after the shove) to have something in common and indeed we find that, while the pucks (each of which, after the shove, moves at its own constant velocity) have speeds that differ from one another (because they have different masses), they all have the same value of the product $mv^2$ and indeed if you put a ½ in front of that product and call it kinetic energy $K$, the common value of $\frac{1}{2}mv^2$ is identical to the product of the magnitude of the force used during the shove, and the distance over which the force is applied. This latter product is what we have defined to be the work $W$ and we recognize that we are dealing with a special case of the work energy principle $W = \Delta K$ , a case in which, for each of the pucks, the initial kinetic energy is zero. We can modify our experiment to obtain more general results, e.g. a smaller constant force over a greater distance results in the same kinetic energy as long as the product of the magnitude of the force and the distance over which it is applied is the same as it was for the other pucks, but it is interesting to consider how different it would seem to us, in the original experiment, as we move from a high-mass puck to a low mass puck. Imagine doing that. You push on the high-mass puck with a certain force, for a certain distance. Now you move on to a low-mass puck. As you push on it from behind, with the same force that you used on the high-mass puck, you notice that the low-mass puck speeds up much more rapidly. You probably find it much more difficult to maintain a steady force because it is simply more difficult to "keep up" with the low mass puck. And of course, it covers the specified distance in a much shorter amount of time. So, although you push it for the same distance, you must push the low-mass puck for a shorter amount of time in order to make it so that both pucks have one and the same kinetic energy. Pondering on it you recognize that if you were to push the low-mass puck for the same amount of *time* as you did the high-mass puck (with the same force), that the low-mass puck would have a greater kinetic energy after the shove, because you would have to push on it over a greater distance, meaning you would have done more work on it. Still, you imagine that if you were to push on each of the pucks for the same amount of time (rather than distance), that their respective motions would have to have something in common, because again, there is something similar about their respective shoves.

## *Now we Move on to Impulse and Momentum*

You decide to do the experiment you have been thinking about. You place each of the pucks at rest on the frictionless surface. You apply one and the same constant force to each of the pucks for one and the same amount of time. Once again, you find this more difficult with the lower mass pucks. While you are pushing on it, a low-mass puck speeds up faster than a high-mass puck does. As a result you have to keep pushing on a low-mass puck over a greater distance and it is going faster when you let it go. Having given all the pucks a similar shove, you expect there to be something about the motion of each of the pucks that is the same as the corresponding characteristic of the motion of all the other pucks. We have already established that the smaller the mass of the puck, the greater the speed, and the greater the kinetic energy of the puck. Experimentally, we find that all the pucks have one and the same value of the product $m\vec{\mathbf{v}}$, where $\vec{\mathbf{v}}$ is the post-shove puck velocity. Further, we find that the value of $m\vec{\mathbf{v}}$ is equal to the product of the constant force $\vec{\mathbf{F}}$ and the time interval $\Delta t$ for which it was applied.

That is,

$$\vec{\mathbf{F}}\Delta t = m\vec{\mathbf{v}}$$

The product of the force and the time interval for which it is applied is such an important quantity that we give it a name, *impulse*, and a symbol $\vec{\mathbf{J}}$.

$$\vec{\mathbf{J}} = \vec{\mathbf{F}}\,\Delta t \qquad (26\text{-}1)$$

Also, as you probably recall from chapter 4, by definition, the product of the mass of an object, and its velocity, is the momentum $\vec{\mathbf{p}}$ of the object.

Thus, the results of the experiment described above can be expressed as

$$\vec{\mathbf{J}} = \vec{\mathbf{p}}$$

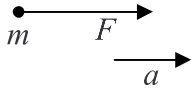
The experiment dealt with a special case, the case in which each object was initially at rest. If we do a similar experiment in which, rather than being initially at rest, each object has some known initial velocity, we find, experimentally, that the impulse is actually equal to the change in momentum.

$$\vec{\mathbf{J}} = \overrightarrow{\Delta\mathbf{p}} \qquad (26\text{-}2)$$

Of course if we start with zero momentum, then the change in momentum *is* the final momentum.

Equation 26-2, $\vec{\mathbf{J}} = \overrightarrow{\Delta\mathbf{p}}$, is referred to as the *Impulse-Momentum Relation*. It is a cause and effect relationship. You apply some impulse (force times time) to an object, and the effect is a change in the momentum of the object. The result, which we have presented as an experimental result, can be derived from Newton's second law of motion. Here we do so for the case in which the force acting on the object is constant during the time interval under consideration. Note that the force which appears in the definition of impulse is the net external force acting on the object. Consider the case of a particle, of mass *m*, which has but one, constant force (which could actually be the vector sum of all the forces) acting on it.

As always, in applying Newton's second law of motion, we start by drawing a free body diagram:



In order to keep track of the vector nature of the quantities involved we apply Newton's 2nd Law in vector form (equation 14-1):

$$\vec{\boldsymbol{a}} = \frac{1}{m}\sum \vec{\mathbf{F}}$$

In the case at hand the sum of the forces is just the one force $\vec{\mathbf{F}}$, so:

$$\vec{\boldsymbol{a}} = \frac{1}{m}\vec{\mathbf{F}}$$

Solving for *F*, we arrive at:

$$\vec{\mathbf{F}} = m\vec{\boldsymbol{a}}$$

multiplying both sides by Δ*t* we obtain

$$\vec{\mathbf{F}}\Delta t = m\vec{\boldsymbol{a}}\Delta t$$

Given that the force is constant, the resulting acceleration is constant. In the case of a constant acceleration, the acceleration can be written as the ratio of the change in $v$ that occurs during the time interval Δ*t*, to the time interval Δ*t* itself.

$$\vec{\boldsymbol{a}} = \frac{\overrightarrow{\Delta \boldsymbol{v}}}{\Delta t}$$

Substituting this into the preceding expression yields:

$$\vec{\mathbf{F}}\Delta t = m\frac{\overrightarrow{\Delta \boldsymbol{v}}}{\Delta t}\Delta t$$

$$\vec{\mathbf{F}}\Delta t = m\overrightarrow{\Delta \boldsymbol{v}}$$

The change in velocity can be expressed as the final velocity $\vec{\boldsymbol{v}}'$ (the velocity at the end of the time interval during which the force acts) minus the initial velocity $\vec{\boldsymbol{v}}$ (the velocity at the start of the time interval): $\overrightarrow{\Delta \boldsymbol{v}} = \vec{\boldsymbol{v}}' - \vec{\boldsymbol{v}}$. Substituting this into $\vec{\mathbf{F}}\Delta t = m\overrightarrow{\Delta \boldsymbol{v}}$ yields

$$\vec{\mathbf{F}}\Delta t = m(\vec{\boldsymbol{v}}' - \vec{\boldsymbol{v}})$$

which can be written as

$$\vec{\mathbf{F}}\Delta t = m\vec{\boldsymbol{v}}' - m\vec{\boldsymbol{v}}$$

Recognizing that $m\vec{v}'$ is the final momentum and that $m\vec{v}$ is the initial momentum we realize that we have

$$\vec{\mathbf{F}}\Delta t = \vec{\mathbf{p}}' - \vec{\mathbf{p}}$$

On the left, we have what is defined to be the impulse, and on the right we have the change in momentum (equation 26-2):

$$\vec{\mathbf{J}} = \overrightarrow{\Delta\mathbf{p}}$$

This completes our derivation of the impulse momentum relation from Newton's 2nd Law.

## *Conservation of Momentum Revisited*

Regarding the conservation of momentum, we first note that, for a particle, if the net external force on the particle is zero, then the impulse, defined by $\vec{\mathbf{J}} = \vec{\mathbf{F}}\Delta t$, delivered to that particle during any time interval $\Delta t$, is 0. If the impulse is zero then from $\vec{\mathbf{J}} = \overrightarrow{\Delta\mathbf{p}}$, the change in momentum must be 0. This means that the momentum $\vec{\mathbf{p}}$ is a constant, and since $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, if the momentum is constant, the velocity must be constant. This result simply confirms that, in the absence of a force, our impulse momentum relation is consistent with Newton's 1st Law of Motion, the one that states that if there is no force on a particle, then the velocity of that particle does not change.

Now consider the case of two particles in which no external forces are exerted on either of the particles. (For a system of two particles, an internal force would be a force exerted by one particle on the other. An external force is a force exerted by something outside the system on something inside the system.) The total momentum of the pair of particles is the vector sum of the momentum of one of the particles and the momentum of the other particle. Suppose that the particles are indeed exerting forces on each other during a time interval $\Delta t$. To keep things simple we will assume that the force that either exerts on the other is constant during the time interval. Let's identify the two particles as particle #1 and particle #2 and designate the force exerted by 1 on 2 as $\vec{\mathbf{F}}_{12}$. Because this force is exerted on particle #2, it will affect the motion of particle #2 and we can write the impulse momentum relation as

$$\vec{\mathbf{F}}_{12}\Delta t = \overrightarrow{\Delta\mathbf{p}}_2 \tag{26-3}$$

Now particle #1 can't exert a force on particle 2 without particle #2 exerting an equal and opposite force back on particle 1. That is, the force $\vec{\mathbf{F}}_{21}$ exerted by particle #2 on particle #1 is the negative of $\vec{\mathbf{F}}_{12}$.

$$\vec{\mathbf{F}}_{21} = -\vec{\mathbf{F}}_{12}$$

Of course $\vec{\mathbf{F}}_{21}$ ("eff of 2 on 1") affects the motion of particle 1 only, and the impulse-momentum relation for particle 1 reads

$$\vec{\mathbf{F}}_{21}\Delta t = \overrightarrow{\Delta\mathbf{p}}_1$$

Replacing $\vec{\mathbf{F}}_{21}$ with $-\vec{\mathbf{F}}_{12}$ we obtain

$$-\mathbf{F}_{12}\Delta t = \overrightarrow{\Delta\mathbf{p}}_1 \qquad (26\text{-}4)$$

Now add equation 26-3 ( $\vec{\mathbf{F}}_{12}\Delta t = \overrightarrow{\Delta\mathbf{p}}_2$ ) and equation 26-4 together. The result is:

$$\begin{aligned}\vec{\mathbf{F}}_{12}\Delta t - \vec{\mathbf{F}}_{12}\Delta t &= \overrightarrow{\Delta\mathbf{p}}_1 + \overrightarrow{\Delta\mathbf{p}}_2 \\ 0 &= \overrightarrow{\Delta\mathbf{p}}_1 + \overrightarrow{\Delta\mathbf{p}}_2\end{aligned}$$

On the right is the total change in momentum for the pair of particles $\overrightarrow{\Delta\mathbf{p}}_{\text{TOTAL}} = \overrightarrow{\Delta\mathbf{p}}_1 + \overrightarrow{\Delta\mathbf{p}}_2$ so what we have found is that

$$0 = \overrightarrow{\Delta\mathbf{p}}_{\text{TOTAL}}$$

which can be written as

$$\overrightarrow{\Delta\mathbf{p}}_{\text{TOTAL}} = 0 \qquad (26\text{-}5)$$

Recapping: If the net external force acting on a pair of particles is zero, the total momentum of the pair of particles does not change. Add a third particle to the mix and any momentum change that it might experience because of forces exerted on it by the original two particles would be canceled by the momentum changes experienced by the other two particles as a result of the interaction forces exerted on them by the third particle. We can extend this to any number of particles, and since objects are made of particles, the concept applies to objects. That is, if, during some time interval, the net external force exerted on a system of objects is zero, then the momentum of that system of objects will not change.

As you should recall from Chapter 4, the concept is referred to as conservation of momentum for the special case in which there is no net transfer of momentum to the system from the surroundings, and you apply it in the case of some physical process such as a collision, by picking a before instant and an after instant, drawing a sketch of the situation at each instant, and writing the fact that, the momentum in the before picture has to be equal to the momentum in the after picture, in equation form: $\vec{\mathbf{p}} = \vec{\mathbf{p}}'$. When you read this chapter, you should again consider yourself responsible for solving any of the problems, and answering any of the questions, that you were responsible for back in Chapter 4.