

### 33 Fluids: Pressure, Density, Archimedes' Principle

*One mistake you see in solutions to submerged-object static fluid problems, is the inclusion, in the free body diagram for the problem, in addition to the buoyant force, of a pressure-times-area force typically expressed as  $F_p = PA$ . This is double counting.*

*Folks that include such a force, in addition to the buoyant force, don't realize that the buoyant force is the net sum of all the pressure-times-area forces exerted, on the submerged object by the fluid in which it is submerged.*

Gases and liquids are fluids. Unlike solids, they flow. A fluid is a liquid or a gas.

#### **Pressure**

A fluid exerts pressure on the surface of any substance with which the fluid is in contact. Pressure is force-per-area. In the case of a fluid in contact with a flat surface over which the pressure of the fluid is constant, the magnitude of the force on that surface is the pressure times the area of the surface. Pressure has units of  $\text{N/m}^2$ .

Never say that pressure is the amount of force exerted on a certain amount of area. Pressure is not an amount of force. Even in the special case in which the pressure over the "certain amount of area" is constant, the pressure is not the amount of force. In such a case, the pressure is what you have to multiply the area by to determine the amount of force.

The fact that the pressure in a fluid is  $5 \text{ N/m}^2$  in no way implies that there is a force of 5 N acting on a square meter of surface (any more than the fact that the speedometer in your car reads 35 mph implies that you are traveling 35 miles or that you have been traveling for an hour). In fact, if you say that the pressure at a particular point underwater in a swimming pool is  $15,000 \text{ N/m}^2$  (fifteen thousand newtons per square meter), you are not specifying any area whatsoever. What you are saying is that any infinitesimal surface element that may be exposed to the fluid at that point will experience an infinitesimal force of magnitude  $dF$  that is equal to  $15,000 \text{ N/m}^2$  times the area  $dA$  of the surface. When we specify a pressure, we're talking about a would-be effect on a would-be surface element.

We talk about an infinitesimal area element because it is entirely possible that the pressure varies with position. If the pressure at one point in a liquid is  $15,000 \text{ N/m}^2$  it could very well be  $16,000 \text{ N/m}^2$  at a point that's less than a millimeter away in one direction and  $14,000 \text{ N/m}^2$  at a point that's less than a millimeter away in another direction.

Let's talk about direction. Pressure itself has no direction. But the force that a fluid exerts on a surface element, because of the pressure of the fluid, does have direction. The force is perpendicular to, and toward, the surface. Isn't that interesting? The direction of the force resulting from some pressure (let's call that the pressure-times-area force) on a surface element is determined by the victim (the surface element) rather than the agent (the fluid).

## Pressure Dependence on Depth

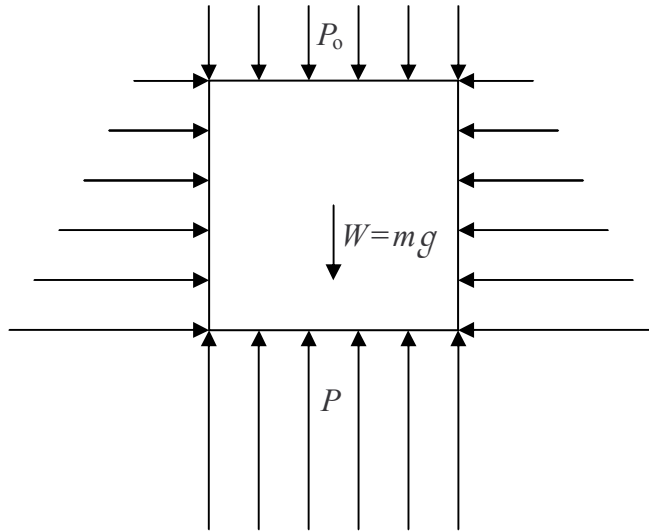
For a fluid near the surface of the earth, the pressure in the fluid increases with depth. You may have noticed this, if you have ever gone deep under water, because you can feel the effect of the pressure on your ear drums. Before we investigate this phenomenon in depth, I need to point out that in the case of a gas, this pressure dependence on depth is, for many practical purposes, negligible. In discussing a container of a gas for instance, we typically state a single value for the pressure of the gas in the container, neglecting the fact that the pressure is greater at the bottom of the container. We neglect this fact because the difference in the pressure at the bottom and the pressure at the top is so very small compared to the pressure itself at the top. We do this when the pressure difference is too small to be relevant, but it should be noted that even a very small pressure difference can be significant. For instance, a helium-filled balloon, released from rest near the surface of the earth would fall to the ground if it weren't for the fact that the air pressure in the vicinity of the lower part of the balloon is greater (albeit only slightly greater) than the air pressure in the vicinity of the upper part of the balloon.

Let's do a thought experiment. (Einstein was fond of thought experiments. They are also called Gedanken experiments. Gedanken is the German word for thought.) Imagine that we construct a pressure gauge as follows: We cap one end of a piece of thin pipe and put a spring completely inside the pipe with one end in contact with the end cap. Now we put a disk whose diameter is equal to the inside diameter of the pipe, in the pipe and bring it into contact with the other end of the spring. We grease the inside walls of the pipe so that the disk can slide freely along the length of the pipe, but we make the fit exact so that no fluid can get past the disk. Now we drill a hole in the end cap, remove all the air from the region of the pipe between the disk and the end cap, and seal up the hole. The position of the disk in the pipe, relative to its position when the spring is neither stretched nor compressed, is directly proportional to the pressure on the outer surface, the side facing away from the spring, of the disk. We calibrate (mark a scale on) the pressure gauge that we have just manufactured, and use it to investigate the pressure in the water of a swimming pool. First we note that, as soon as we removed the air, the gauge started to indicate a significant pressure (around  $1.013 \times 10^5 \text{ N/m}^2$ ), namely the air pressure in the atmosphere. Now we move the gauge around and watch the gauge reading. Wherever we put the gauge (we define the location of the gauge to be the position of the center point on the outer surface of the disk) on the surface of the water, we get one and the same reading, (the air pressure reading). Next we verify that the pressure reading does indeed increase as we lower the gauge deeper and deeper into the water. Then we find, the point I wrote this paragraph to make, that if we move the gauge around horizontally at one particular depth, the pressure reading does not change. That's the experimental result I want to use in the following development, the experimental fact that the pressure has one and the same value at all points that are at one and the same depth in a fluid.

Here we derive a formula that gives the pressure in an incompressible static fluid as a function of the depth in the fluid. Let's get back into the swimming pool. Now imagine a closed surface enclosing a volume, a region in space, that is full of water. I'm going to call the water in such a volume, "a volume of water," and I'm going to give it another name as well. If it were ice, I would call it a chunk of ice, but since it is liquid water, I'll call it a "slug" of water. We're going

to derive the pressure vs. depth relation by investigating the equilibrium of an “object” which is a slug of water.

Consider a cylindrical slug of water whose top is part of the surface of the swimming pool and whose bottom is at some arbitrary depth  $h$  below the surface. I'm going to draw the slug here, isolated from its surroundings. The slug itself is, of course, surrounded by the rest of the water in the pool.



In the diagram, we use arrows to convey the fact that there is pressure-times-area force on every element of the surface of the slug. Now the downward pressure-times-area force on the top of the slug is easy to express in terms of the pressure because the pressure on every infinitesimal area element making up the top of the slug has one and the same value. In terms of the determination of the pressure-times-area, this is the easy case. The magnitude of the force,  $F_0$ , is just the pressure  $P_0$  times the area  $A$  of the top of the cylinder.

$$F_0 = P_0 A$$

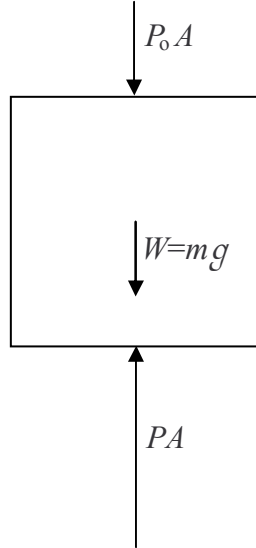
A similar argument can be made for the bottom of the cylinder. All points on the bottom of the cylinder are at the same depth in the water so all points are at one and the same pressure  $P$ . The bottom of the cylinder has the same area  $A$  as the top so the magnitude of the upward force  $F$  on the bottom of the cylinder is given by

$$F = PA$$

As to the sides, if we divide the sidewalls of the cylinder up into an infinite set of equal-sized infinitesimal area elements, for every sidewall area element, there is a corresponding area element on the opposite side of the cylinder. The pressure is the same on both elements because they are at the same depth. The two forces then have the same magnitude, but because the elements face in opposite directions, the forces have opposite directions. Two opposite but equal

forces add up to zero. In such a manner, all the forces on the sidewall area elements cancel each other out.

Now we are in a position to draw a free body diagram of the cylindrical slug of water.



Applying the equilibrium condition

$$\sum F_{\uparrow} = 0$$

yields

$$PA - mg - P_0 A = 0 \quad (33-1)$$

At this point in our derivation of the relation between pressure and depth, the depth does not explicitly appear in the equation. The mass of the slug of water, however, does depend on the length of the slug which is indeed the depth  $h$ . First we note that

$$m = \rho V \quad (33-2)$$

where  $\rho$  is the density, the mass-per-volume, of the water making up the slug and  $V$  is the volume of the slug. The volume of a cylinder is its height times its face area so we can write

$$m = \rho h A$$

Substituting this expression for the mass of the slug into equation 33-1 yields

$$PA - \rho h A g - P_0 A = 0$$

$$P - \rho h g - P_0 = 0$$

$$P = P_0 + \rho g h \quad (33-3)$$

While we have been writing specifically about water, the only thing in the analysis that depends on the identity of the incompressible fluid is the density  $\rho$ . Hence, as long as we use the density of the fluid in question, equation 33-3 ( $P = P_0 + \rho gh$ ) applies to any incompressible fluid. It says that the pressure at any depth  $h$  is the pressure at the surface plus  $\rho gh$ .

A few words on the units of pressure are in order. We have stated that the units of pressure are  $\text{N}/\text{m}^2$ . This combination of units is given a name. It is called the *pascal*, abbreviated Pa.

$$1 \text{ Pa} = 1 \frac{\text{N}}{\text{m}^2}$$

Pressures are often quoted in terms of the non-SI unit of pressure, the *atmosphere*, abbreviated atm and defined such that, on the average, the pressure of the earth's atmosphere at sea level is 1 atm. In terms of the pascal,

$$1 \text{ atm} = 1.013 \times 10^5 \text{ Pa}$$

The big mistake that folks make in applying equation 33-3 ( $P = P_0 + \rho gh$ ) is to ignore the units. They'll use 1 atm for  $P_0$  and without converting that to pascals, they'll add the product  $\rho gh$  to it. Of course, if one uses SI units for  $\rho$ ,  $g$ , and  $h$ , the product  $\rho gh$  comes out in  $\text{N}/\text{m}^2$  which is a pascal which is definitely not an atmosphere (but rather, about a hundred-thousandth of an atmosphere). Of course one can't add a value in pascals to a value in atmospheres. The way to go is to convert the value of  $P_0$  that was given to you in units of atmospheres, to pascals, and then add the product  $\rho gh$  (in SI units) to your result so that your final answer comes out in pascals.

## Gauge Pressure

Remember the gauge we constructed for our thought experiment? That part about evacuating the inside of the pipe presents quite the manufacturing challenge. The gauge would become inaccurate as air leaked in by the disk. As regards function, the description is fairly realistic in terms of actual pressure gauges in use, except for the pumping of the air out the pipe. To make it more like an actual gauge that one might purchase, we would have to leave the interior open to the atmosphere. In use then, the gauge reads zero when the pressure on the sensor end is 1 atmosphere, and in general, indicates the amount by which the pressure being measured exceeds atmospheric pressure. This quantity, the amount by which a pressure exceeds atmospheric pressure, is called gauge pressure (since it is the value registered by a typical pressure gauge.) When it needs to be contrasted with gauge pressure, the actual pressure that we

have been discussing up to this point is called *absolute pressure*. The absolute pressure and the gauge pressure are related by:

$$P = P_G + P_O \quad (33-4)$$

where:

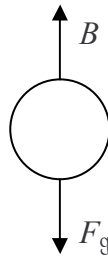
$P$  is the absolute pressure,  
 $P_G$  is the gauge pressure, and  
 $P_O$  is atmospheric pressure.

When you hear a value of pressure (other than the so-called barometric pressure of the earth's atmosphere) in your everyday life, it is typically a gauge pressure (even though one does not use the adjective "gauge" in discussing it.) For instance, if you hear that the recommended tire pressure for your tires is 32 psi (pounds per square inch) what is being quoted is a gauge pressure. Folks that work on ventilation systems often speak of negative air pressure. Again, they are actually talking about gauge pressure, and a negative value of gauge pressure in a ventilation line just means that the absolute pressure is less than atmospheric pressure.

### **Archimedes' Principle**

The net pressure-times-area force on an object submerged in a fluid, the vector sum of the forces on all the infinite number of infinitesimal surface area elements making up the surface of an object, is *upward* because of the fact that pressure increases with depth. The upward pressure-times-area force on the bottom of an object is greater than the downward pressure-times-area force on the top of the object. The result is a net upward force on any object that is either partly or totally submerged in a fluid. The force is called the buoyant force on the object. The agent of the buoyant force is the fluid.

If you take an object in your hand, submerge the object in still water, and release the object from rest, one of three things will happen: The object will experience an upward acceleration and bob to the surface, the object will remain at rest, or the object will experience a downward acceleration and sink. We have emphasized that the buoyant force is always upward. So why on earth would the object ever sink? The reason is, of course, that after you release the object, the buoyant force is not the only force acting on the object. The gravitational force still acts on the object when the object is submerged. Recall that the earth's gravitational field permeates everything. For an object that is touching nothing of substance but the fluid it is in, the free body diagram (without the acceleration vector being included) is always the same (except for the relative lengths of the arrows):

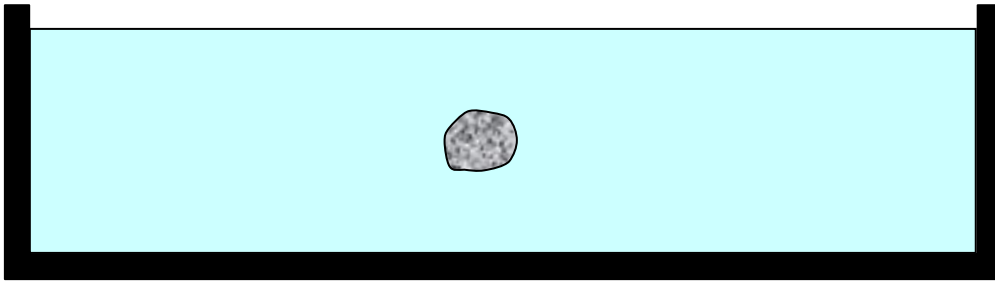


and the whole question as to whether the object (released from rest in the fluid) sinks, stays put, or bobs to the surface, is determined by how the magnitude of the buoyant force compares with

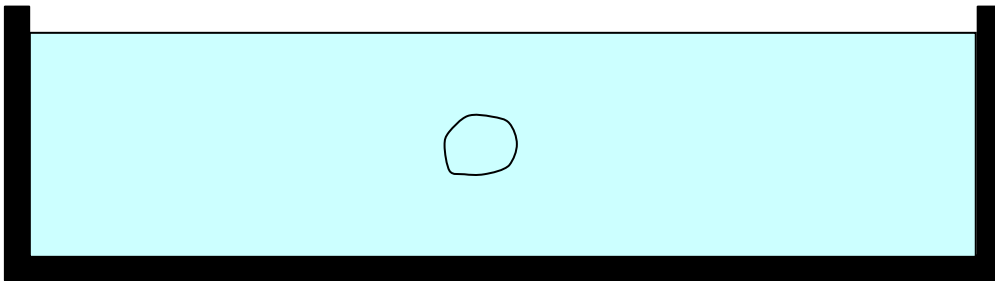
that of the gravitational force. If the buoyant force is greater, the net force is upward and the object bobs toward the surface. If the buoyant force and the gravitational force are equal in magnitude, the object stays put. And if the gravitational force is greater, the object sinks.

So how does one determine how big the buoyant force on an object is? First, the trivial case: If the only forces on the object are the buoyant force and the gravitational force, and the object remains at rest, then the buoyant force must be equal in magnitude to the gravitational force. This is the case for an object such as a boat or a log which is floating on the surface of the fluid it is in.

But suppose the object is not freely floating at rest. Consider an object that is submerged in a fluid. We have no information on the acceleration of the object, but we cannot assume it to be zero. Assume that a person has, while maintaining a firm grasp on the object, submerged the object in fluid, and then, released it from rest. We don't know which way it is going from there, but we can *not* assume that it is going to stay put.



To derive our expression for the buoyant force, we do a little thought experiment. Imagine replacing the object with a slug of fluid (the same kind of fluid as that in which the object is submerged), where the slug of fluid has the exact same size and shape as the object.



From our experience with still water we know that the slug of fluid would indeed stay put, meaning that *it* is in equilibrium.

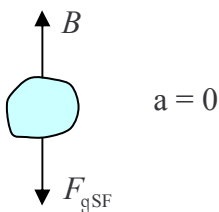


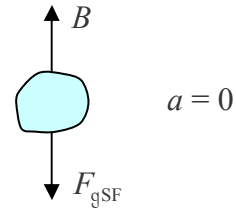
Table of Forces			
<i>Symbol=?</i>	<i>Name</i>	<i>Agent</i>	<i>Victim</i>
$B$	Buoyant Force	The Surrounding Fluid	The Slug of Fluid
$F_{gSF} = m_{SF}g$	Gravitational Force on the Slug of Fluid	The Earth's Gravitational Field	The Slug of Fluid

Applying the equilibrium equation  $\sum F_{\uparrow} = 0$  to the slug of fluid yields:

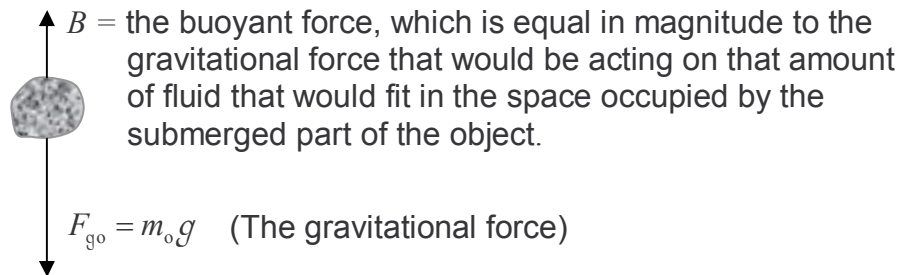
$$\sum F_{\uparrow} = 0$$

$$B - F_{\text{gSF}} = 0$$

$$B = F_{\text{gSF}}$$



The last equation states that the buoyant force on the slug of fluid is equal to the gravitational force on the slug of fluid. Now get this; this is the crux of the derivation: Because the slug of fluid has the exact same size and shape as the original object, it presents the exact same surface to the surrounding fluid, and hence, the surrounding fluid exerts the same buoyant force on the slug of fluid as it does on the original object. Since the buoyant force on the slug of fluid is equal in magnitude to the gravitational force acting on the slug of fluid, the buoyant force on the original object is equal in magnitude to the gravitational force acting on the slug of fluid. This is Archimedes' principle.



Archimedes' Principle states that: The buoyant force on an object that is either partly or totally submerged in a fluid is upward, and is equal in magnitude to the gravitational force that would be acting on that amount of fluid that would be where the object is if the object wasn't there. For an object that is totally submerged, the *volume* of that amount of fluid that would be where the object is if the object wasn't there is equal to the volume of the object itself. But for an object that is only partly submerged, the volume of that amount of fluid that would be where the object is if the object wasn't there is equal to the (typically unknown) volume of the *submerged part* of the object. However, if the object is freely floating at rest, the equilibrium equation (instead of Archimedes' Principle) can be used to quickly establish that the buoyant force (of a freely floating object such as a boat) is equal in magnitude to the gravitational force acting on the object itself.