

34 Pascal's Principle, the Continuity Equation, and Bernoulli's Principle

There are a couple of mistakes that tend to crop up with some regularity in the application of the Bernoulli equation $P + \frac{1}{2}\rho v^2 + \rho gh = \text{constant}$. Firstly, folks tend to forget to create a diagram in order to identify point 1 and point 2 in the diagram so that they can write the Bernoulli equation in its useful form: $P_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2$. Secondly, when both the velocities in Bernoulli's equation are unknown, they forget that there is another equation that relates the velocities, namely, the continuity equation in the form $A_1 v_1 = A_2 v_2$ which states that the flow rate at position 1 is equal to the flow rate at position 2.

Pascal's Principle

Experimentally, we find that if you increase the pressure by some given amount at one location in a fluid, the pressure increases by that same amount everywhere in the fluid. This experimental result is known as Pascal's Principle.

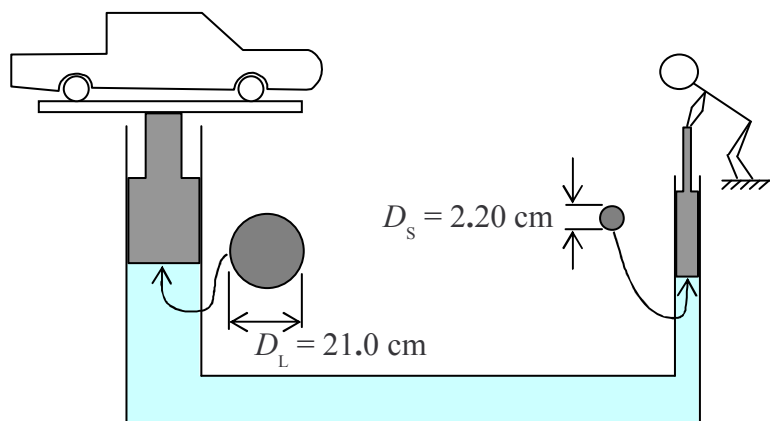
We take advantage of Pascal's principle every time we step on the brakes of our cars and trucks. The brake system is a hydraulic system. The fluid is oil that is called hydraulic fluid. When you depress the brake pedal you increase the pressure everywhere in the fluid in the hydraulic line. At the wheels, the increased pressure acting on pistons attached to the brake pads pushes them against disks or drums connected to the wheels.

Example 34-1

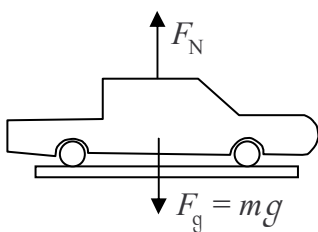
A simple hydraulic lift consists of two pistons, one larger than the other, in cylinders connected by a pipe. The cylinders and pipe are filled with water. In use, a person pushes down upon the smaller piston and the water pushes upward on the larger piston. The diameter of the smaller piston is 2.20 centimeters. The diameter of the larger piston is 21.0 centimeters. On top of the larger piston is a metal support and on top of that is a car. The combined mass of the support-plus-car is 998 kg. Find the force that the person must exert on the smaller piston to raise the car at a constant velocity. Neglect the masses of the pistons.

Solution

We start our solution with a sketch.



Now, let's find the force R_N exerted on the larger piston by the car support. By Newton's 3rd Law, it is the same as the normal force F_N exerted *by* the larger piston *on* the car support. We'll draw and analyze the free body diagram of the car-plus-support to get that.



$$\sum F_{\uparrow} = 0$$

$$F_N - mg = 0$$

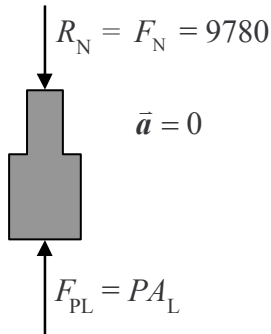
$$F_N = mg$$

$$F_N = 998 \text{ kg} \left(9.80 \frac{\text{newtons}}{\text{kg}} \right)$$

$$F_N = 9780 \text{ newtons}$$

Table of Forces			
<i>Symbol</i>	<i>Name</i>	<i>Agent</i>	<i>Victim</i>
$F_g = mg$	Gravitational Force on Support-Plus-Car	The Earth's Gravitational Field	The Support-Plus-Car
F_N	Normal Force	The Large Piston	The Support-Plus-Car

Now we analyze the equilibrium of the larger piston to determine what the pressure in the fluid must be in order for the fluid to exert enough force on the piston (with the car-plus-support on it) to keep it moving at constant velocity.



$$\sum F_{\uparrow} = 0$$

$$F_{\text{PL}} - R_{\text{N}} = 0$$

$$PA_{\text{L}} - R_{\text{N}} = 0$$

$$P = \frac{R_{\text{N}}}{A_{\text{L}}}$$

(34-1)

A_{L} is the area of the face of the larger piston. We can use the given larger piston diameter $D_{\text{L}} = 0.210 \text{ m}$ to determine the area of the face of the larger piston as follows:

$$A_{\text{L}} = \pi r_{\text{L}}^2 \quad \text{where } r_{\text{L}} = \frac{D_{\text{L}}}{2} \text{ is the radius of the larger piston.}$$

$$A_{\text{L}} = \pi \left(\frac{D_{\text{L}}}{2} \right)^2$$

$$A_{\text{L}} = \pi \left(\frac{0.210 \text{ m}}{2} \right)^2$$

$$A_{\text{L}} = 0.03464 \text{ m}^2$$

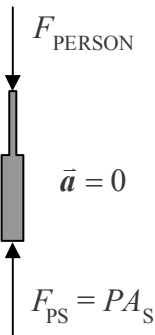
Substituting this and the value $R_{\text{N}} = F_{\text{N}} = 9780 \text{ newtons}$ into equation 34-1 above yields

$$P = \frac{9780 \text{ newtons}}{0.03464 \text{ m}^2}$$

$$P = 282333 \frac{\text{N}}{\text{m}^2} \quad (\text{We intentionally keep 3 too many significant figures in this } \textit{intermediate} \text{ result.})$$

Table of Forces			
<i>Symbol</i>	<i>Name</i>	<i>Agent</i>	<i>Victim</i>
$R_{\text{N}} = F_{\text{N}} = 9780 \text{ newtons}$	Interaction Partner to Normal Force (see above)	Support (That part, of the hydraulic lift, that the car is on.)	Large Piston
F_{PL}	Pressure-Related Force on Large Piston	The Water	Large Piston

Now we just have to analyze the equilibrium of the smaller piston to determine the force that the person must exert on the smaller piston.



$$\sum F_{\uparrow} = 0$$

$$F_{PS} - F_{PERSON} = 0$$

$$PA_S - F_{PERSON} = 0$$

$$F_{PERSON} = PA_S$$

The area A_S of the face of the smaller piston is just pi times the square of the radius of the smaller piston where the radius is $\frac{D_S}{2}$, half the diameter of the smaller piston. So:

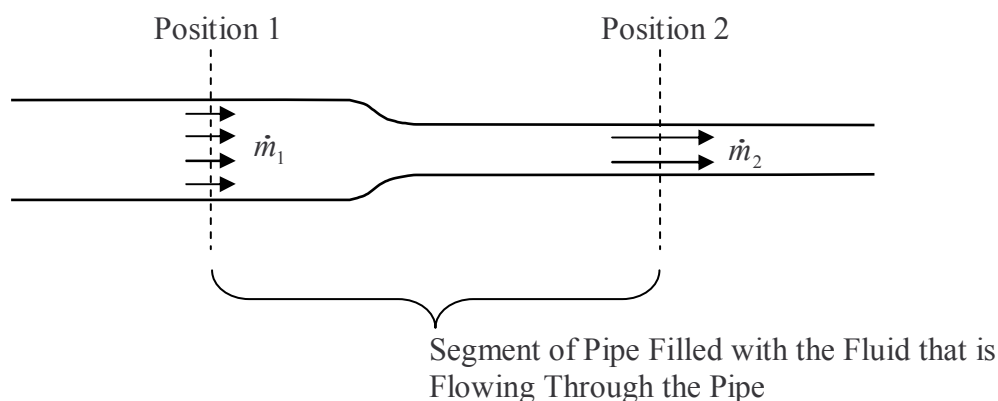
$$F_{PERSON} = P\pi\left(\frac{D_S}{2}\right)^2$$

$$F_{PERSON} = 282333 \frac{\text{N}}{\text{m}^2} \pi \left(\frac{0.0220 \text{ m}}{2}\right)^2$$

$$F_{PERSON} = 107 \text{ N}$$

Fluid in Motion—the Continuity Principle

The Continuity Principle is a fancy name for something that common sense will tell you has to be the case. It is simply a statement of the fact that for any section of a single pipe, filled with an incompressible fluid (an idealization approached by liquids), through which the fluid with which the pipe is filled is flowing, the amount of fluid that goes in one end in any specified amount of time is equal to the amount that comes out the other end in the same amount of time. If we quantify the amount of fluid in terms of the mass, this is a statement of the conservation of mass. Having stipulated that the segment is filled with fluid, the incoming fluid has no room to expand in the segment. Having stipulated that the fluid is incompressible, the molecules making up the fluid cannot be packed closer together; that is, the density of the fluid cannot change. With these stipulations, the total mass of the fluid in the segment of pipe cannot change, so, any time a certain mass of the fluid flows in one end of the segment, the same mass of the fluid must flow out the other end.

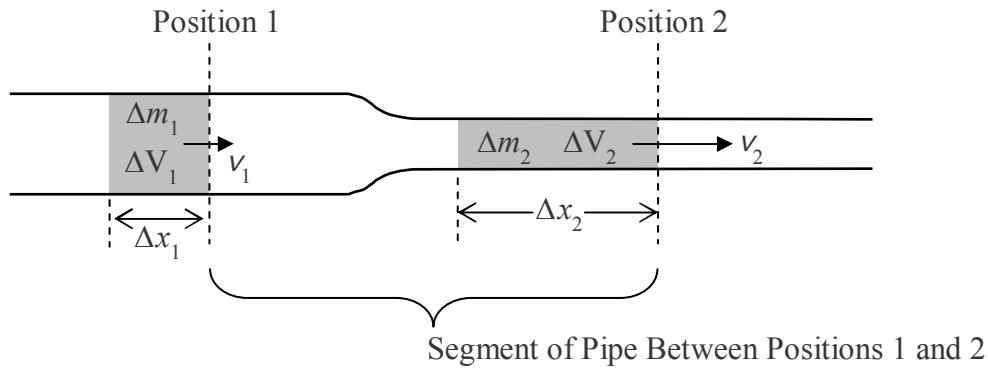


This can only be the case if the mass flow rate, the number of kilograms-per-second passing a given position in the pipe, is the same at both ends of the pipe segment.

$$\dot{m}_1 = \dot{m}_2 \quad (34-2)$$

An interesting consequence of the continuity principle is the fact that, in order for the mass flow rate (the number of kilograms per second passing a given position in the pipe) to be the same in a fat part of the pipe as it is in a skinny part of the pipe, the velocity of the fluid (i.e. the velocity of the molecules of the fluid) must be greater in the skinny part of the pipe. Let's see why this is the case.

Here, we again depict a pipe in which an incompressible fluid is flowing.



Keeping in mind that the entire pipe is filled with the fluid, the shaded region on the left represents the fluid that will flow past position 1 in time Δt and the shaded region on the right represents the fluid that will flow past position 2 in the same time Δt . In both cases, in order for the entire slug of fluid to cross the relevant position line, the slug must travel a distance equal to its length. Now the slug labeled Δm_2 has to be longer than the slug labeled Δm_1 since the pipe is skinnier at position 2 and by the continuity equation $\Delta m_1 = \Delta m_2$ (the amount of fluid that flows into the segment of the pipe between position 1 and position 2 is equal to the amount of fluid that flows out of it). So, if the slug at position 2 is longer and it has to travel past the position line in the same amount of time as it takes for the slug at position 1 to travel past its position line, the fluid velocity at position 2 must be greater. The fluid velocity is greater at a skinnier position in the pipe.

Let's get a quantitative relation between the velocity at position 1 and the velocity at position 2. Starting with

$$\Delta m_1 = \Delta m_2$$

we use the definition of density to replace each mass with the density of the fluid times the relevant volume:

$$\rho \Delta V_1 = \rho \Delta V_2$$

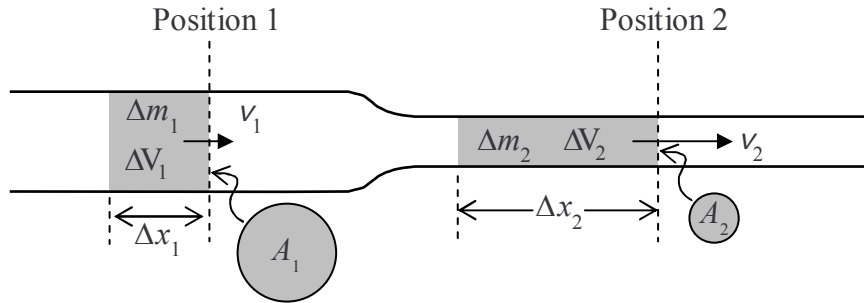
Dividing both sides by the density tells us something you already know:

$$\Delta V_1 = \Delta V_2$$

As an aside, we note that if you divide both sides by Δt and take the limit as Δt goes to zero, we have $\dot{V}_1 = \dot{V}_2$ which is an expression of the continuity principle in terms of volume flow rate. The volume flow rate is typically referred to simply as the *flow rate*.

While we use the SI units $\frac{\text{m}^3}{\text{s}}$ for flow rate, the reader may be more familiar with flow rate expressed in units of gallons per minute.

Now back to our goal of finding a mathematical relation between the velocities of the fluid at the two positions in the pipe. Here we copy the diagram of the pipe and add, to the copy, a depiction of the face of slug 1 of area A_1 and the face of slug 2 of area A_2 .



We left off with the fact that $\Delta V_1 = \Delta V_2$. Each volume can be replaced with the area of the face of the corresponding slug times the length of that slug. So,

$$A_1 \Delta x_1 = A_2 \Delta x_2$$

Recall that Δx_1 is not only the length of slug 1, it is also how far slug 1 must travel in order for the entire slug of fluid to get past the position 1 line. The same is true for slug 2 and position 2. Dividing both sides by the one time interval Δt yields:

$$A_1 \frac{\Delta x_1}{\Delta t} = A_2 \frac{\Delta x_2}{\Delta t}$$

Taking the limit as Δt goes to zero results in:

$$A_1 v_1 = A_2 v_2 \tag{34-3}$$

This is the relation, between the velocities, that we have been looking for. It applies to any pair of positions in a pipe completely filled with an incompressible fluid. It can be written as

$$Av = \text{constant} \tag{34-4}$$

which means that the product of the cross-sectional area of the pipe and the velocity of the fluid at that cross section is the same for every position along the fluid-filled pipe. To take advantage of this fact, one typically writes, in equation form, that the product Av at one location is equal to the same product at another location. In other words, one writes equation 34-3.

Note that the expression Av , the product of the cross-sectional area of the pipe, at a particular position, and the velocity of the fluid at that same position, having been derived by dividing an expression for the volume of fluid ΔV that would flow past a given position of the pipe in time Δt , by Δt , and taking the limit as Δt goes to zero, is none other than the *flow rate* (the volume flow rate) discussed in the aside above.

$$\text{Flow Rate} = Av$$

Further note that if we multiply the flow rate by the density of the fluid, we get the mass flow rate.

$$\dot{m} = \rho Av \quad (34-5)$$

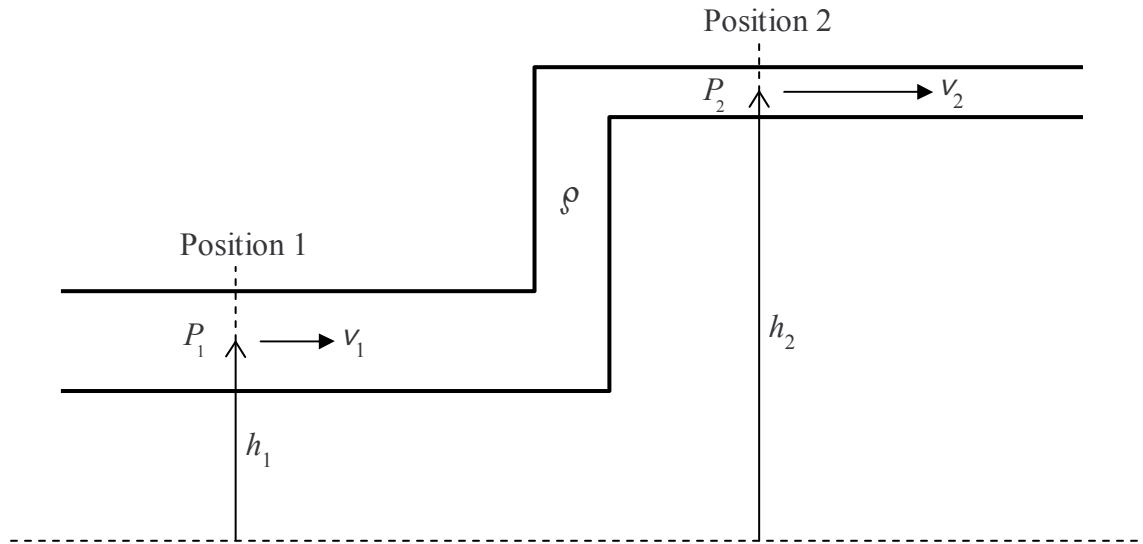
Fluid in Motion—Bernoulli's Principle

The derivation of Bernoulli's Equation represents an elegant application of the Work-Energy Theorem. Here we discuss the conditions under which Bernoulli's Equation applies and then simply state and discuss the result.

Bernoulli's Equation applies to a fluid flowing through a full pipe. The degree to which Bernoulli's Equation is accurate depends on the degree to which the following conditions are met:

- 1) The fluid must be experiencing steady state flow. This means that the flow rate at all positions in the pipe is not changing with time.
- 2) The fluid must be experiencing streamline flow. Pick any point in the fluid. The infinitesimal fluid element at that point, at an instant in time, traveled along a certain path to arrive at that point in the fluid. In the case of streamline flow, every infinitesimal element of fluid that ever finds itself at that same point traveled the same path. (Streamline flow is the opposite of turbulent flow.)
- 3) The fluid must be non-viscous. This means that the fluid has no tendency to "stick to" either the sides of the pipe or to itself. (Molasses has high viscosity. Alcohol has low viscosity.)

Consider a pipe full of a fluid that is flowing through the pipe. In the most general case, the cross-sectional area of the pipe is not the same at all positions along the pipe and different parts of the pipe are at different elevations relative to an arbitrary, but fixed, reference level.



Pick any two positions along the pipe, e.g. positions 1 and 2 in the diagram above. (You already know that, in accord with the continuity principle, $A_1v_1 = A_2v_2$.) Consider the following unnamed sum of terms:

$$P + \frac{1}{2}\rho v^2 + \rho gh$$

where, at the position under consideration:

P is the pressure of the fluid,

ρ (the Greek letter rho) is the density of the fluid,

v is the magnitude of the velocity of the fluid,

$g = 9.80 \frac{\text{N}}{\text{kg}}$ is the near-surface magnitude of the earth's gravitational field, and

h is the elevation, relative to a fixed reference level, of the position in the pipe.

The Bernoulli Principle states that this unnamed sum of terms has the same value at each and every position along the pipe. Bernoulli's equation is typically written:

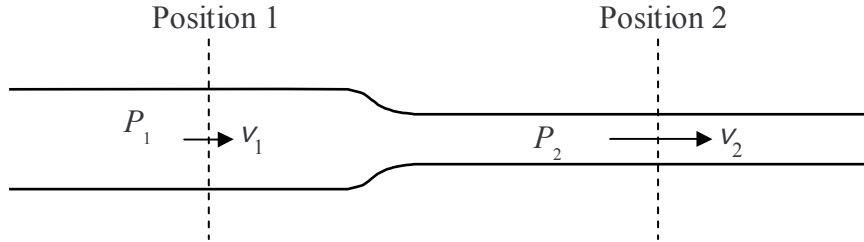
$$P + \frac{1}{2}\rho v^2 + \rho gh = \text{constant} \quad (34-6)$$

but to use it, you have to pick two positions along the pipe and write an equation stating that the

value of the unnamed sum of terms is the same at one of the positions as it is at the other.

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2 \quad (34-7)$$

A particularly interesting characteristic of fluids is incorporated in this equation. Suppose positions 1 and 2 are at one and the same elevation as depicted in the following diagram:



Then $h_1 = h_2$ in equation 34-7 and equation 34-7 becomes:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Check it out. If $v_2 > v_1$ then P_2 must be less than P_1 in order for the equality to hold. This equation is saying that, where the velocity of the fluid is high, the pressure is low.